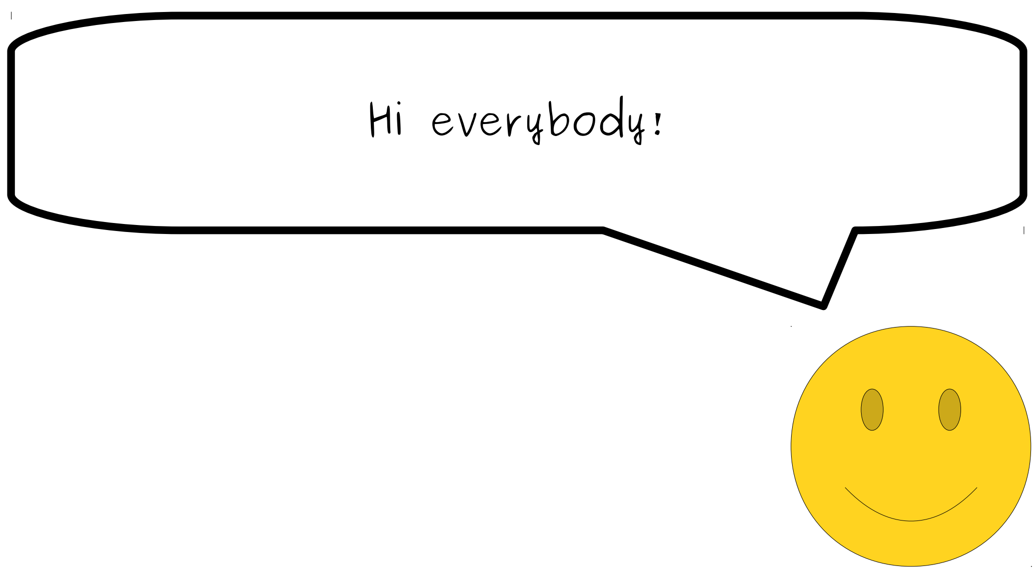
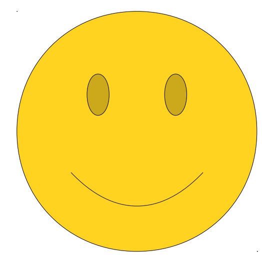
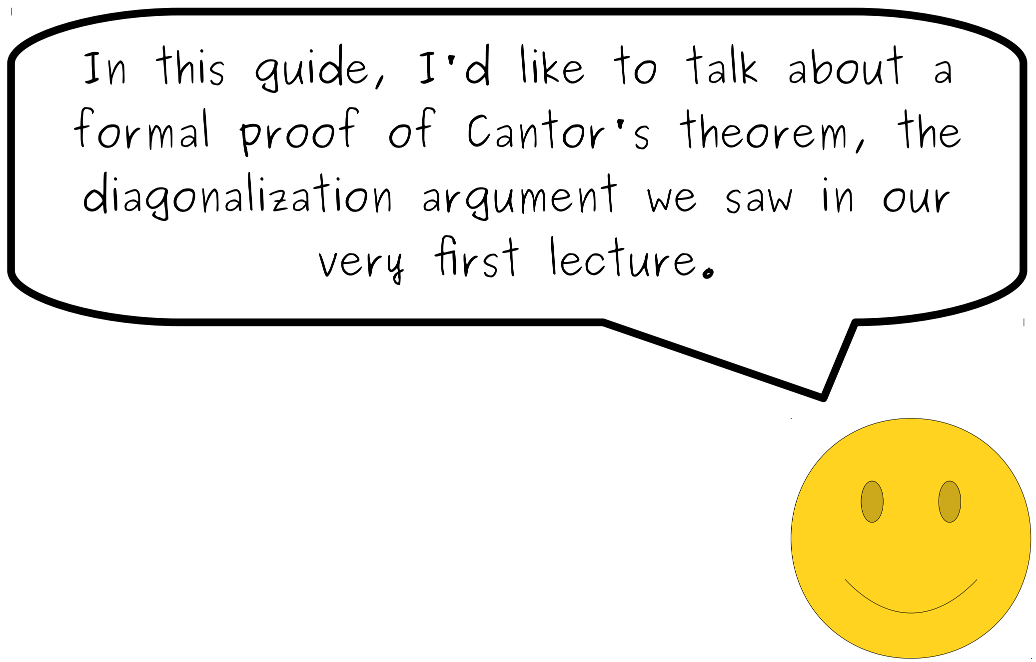
Guide to Cantor's Theorem

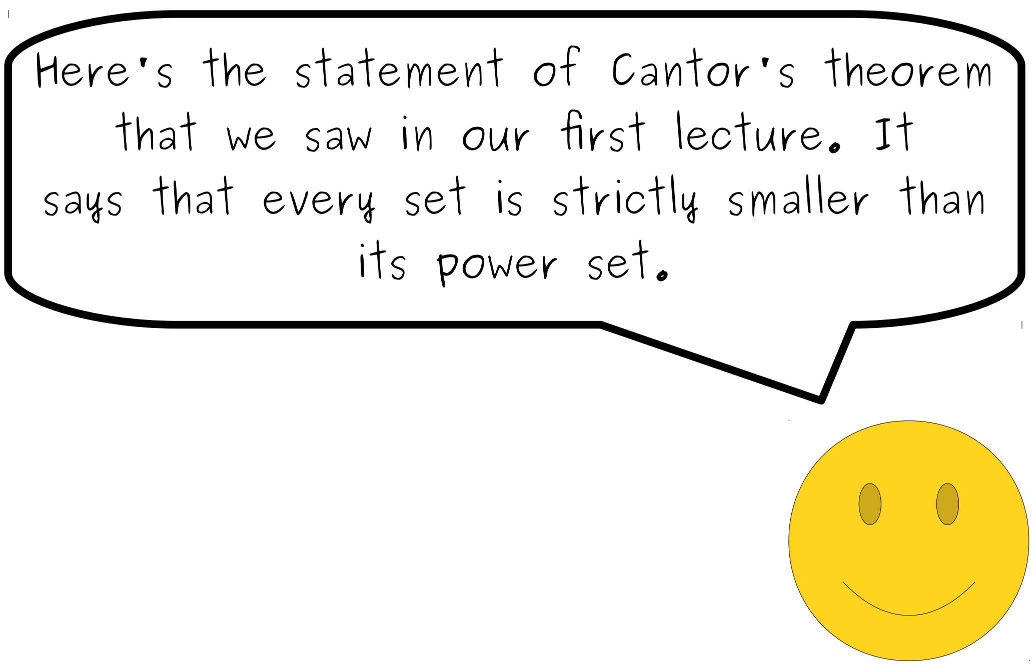

Hi everybody!

In Monday's lecture, we covered a proof by diagonalization where we proved that $|\mathbb{N}| \neq |\mathbb{R}|$.




In this guide, I'd like to talk about a formal proof of Cantor's theorem, the diagonalization argument we saw in our very first lecture.

If S is a set, then $|S| < |\wp(S)|$



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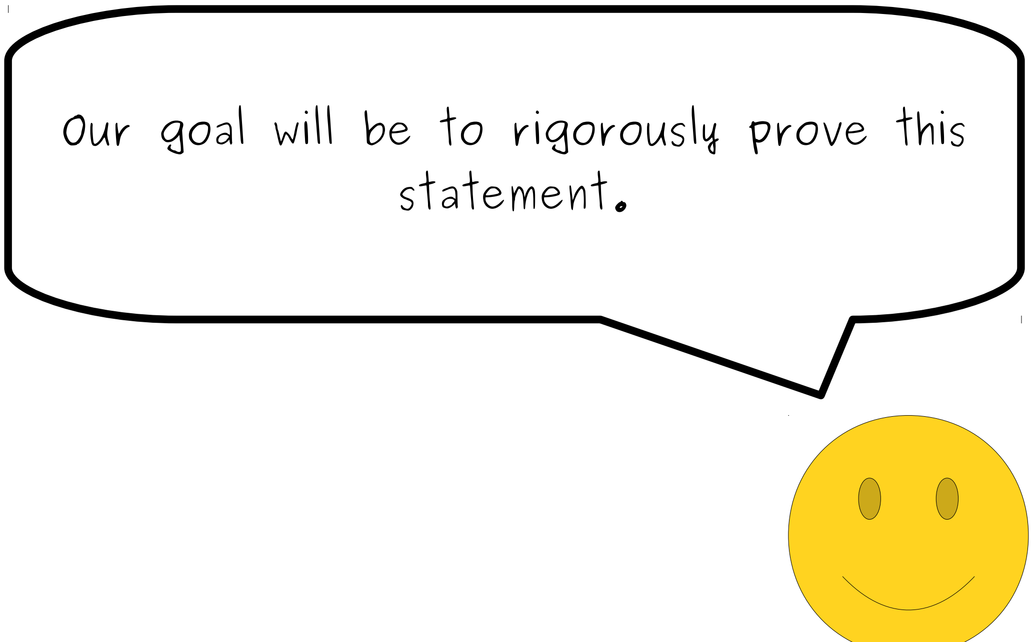

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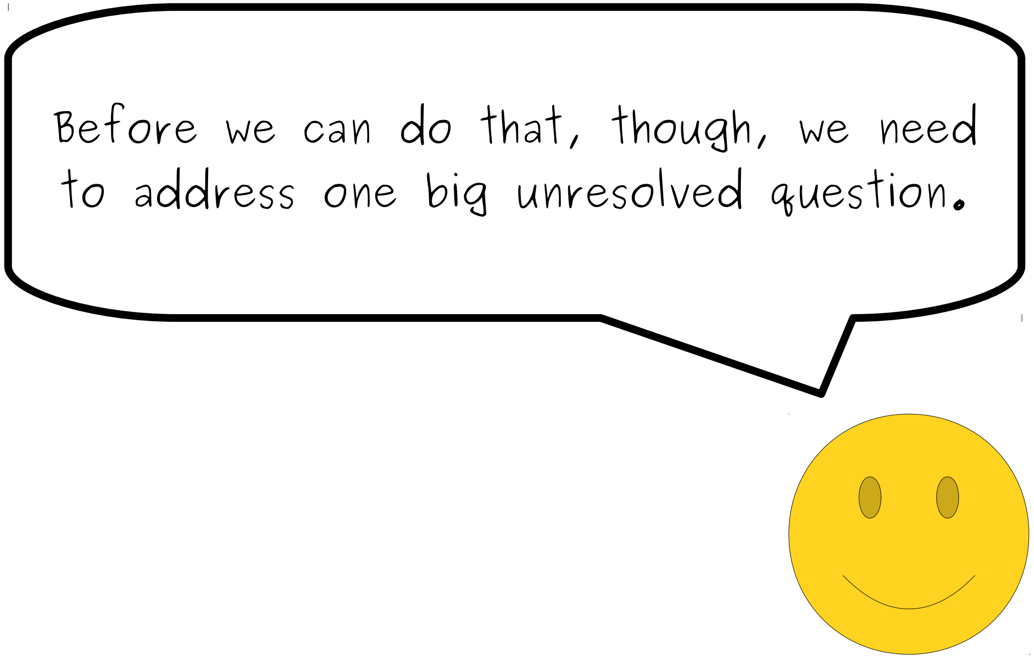

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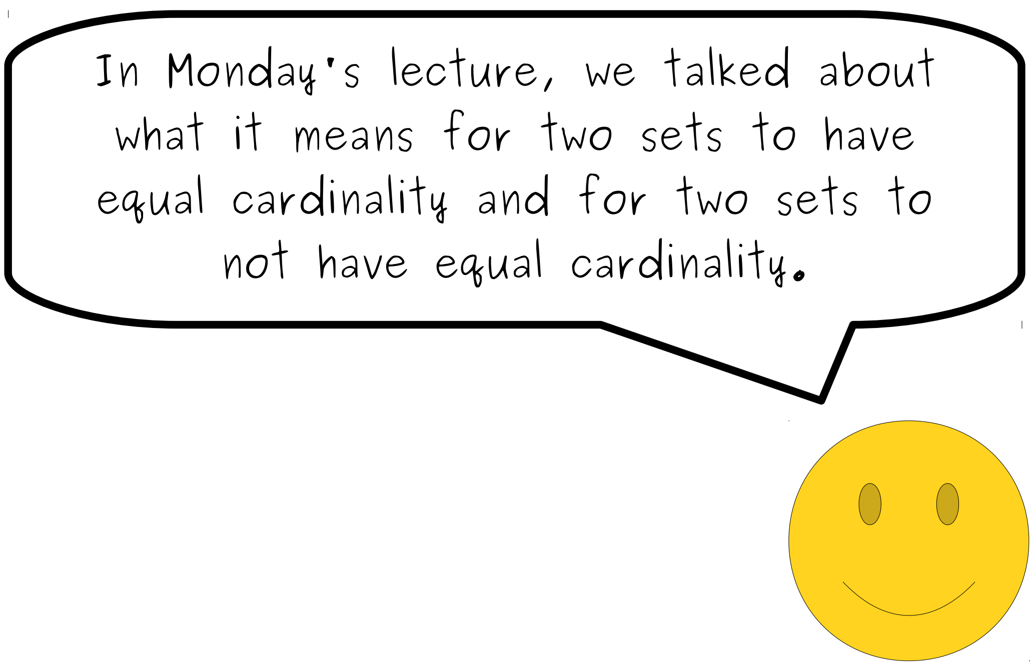

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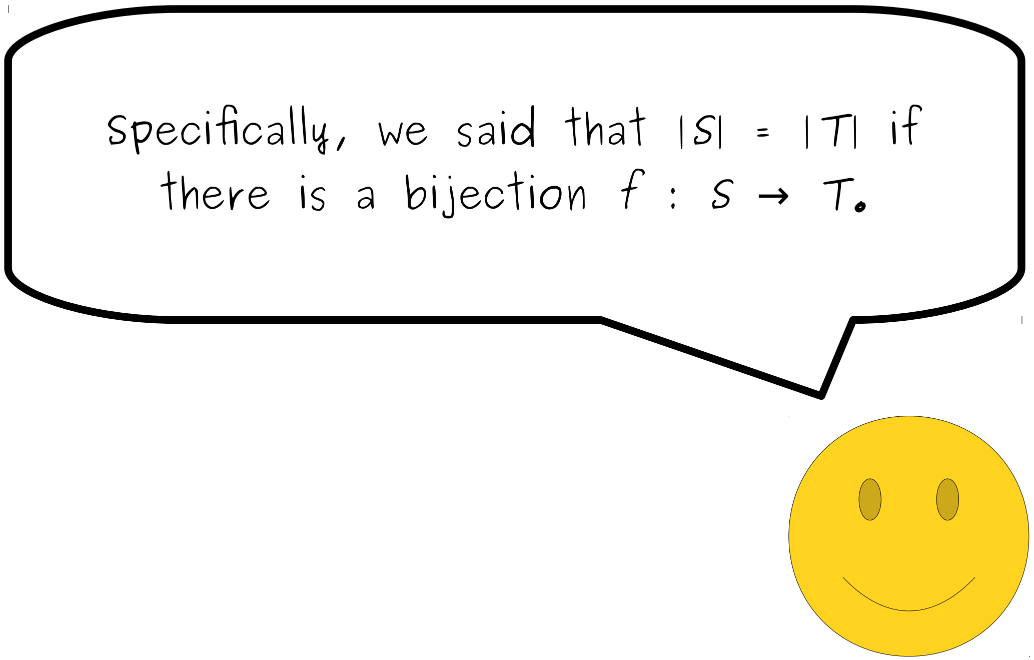

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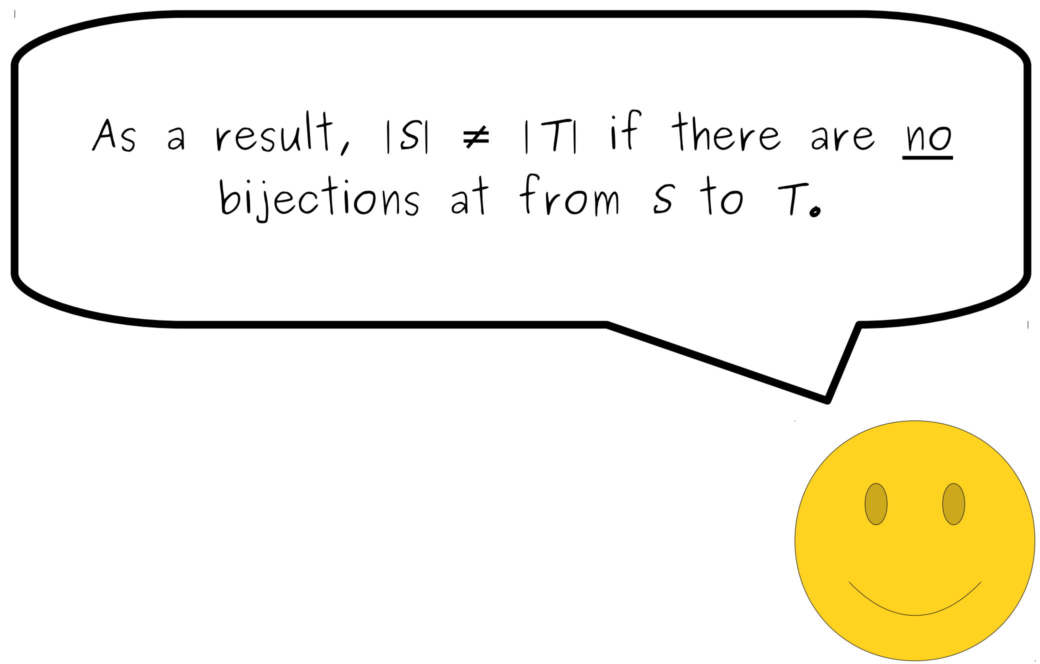

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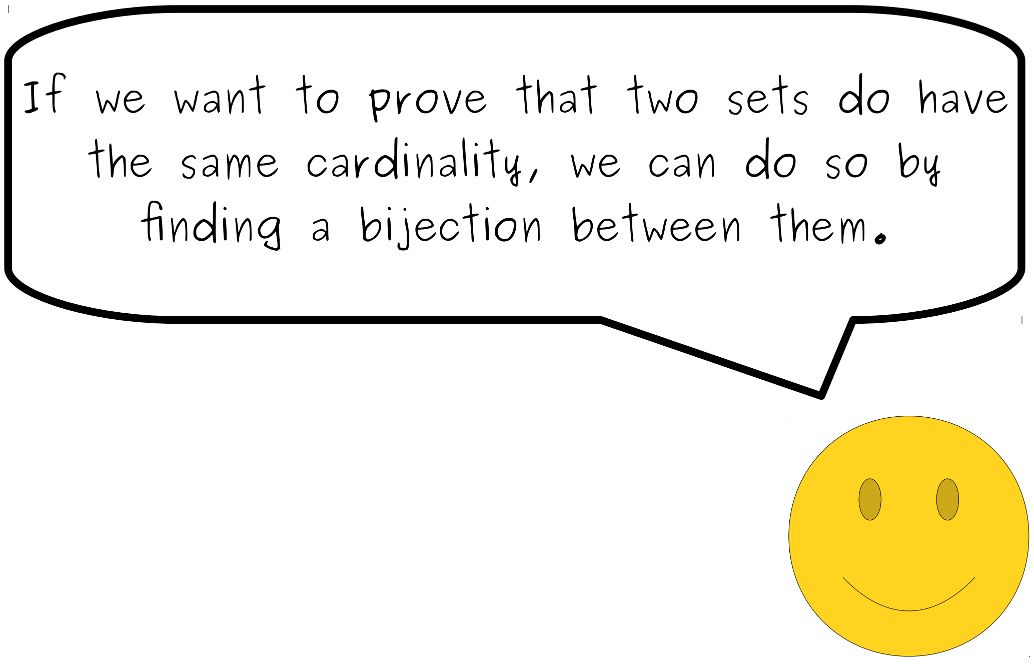

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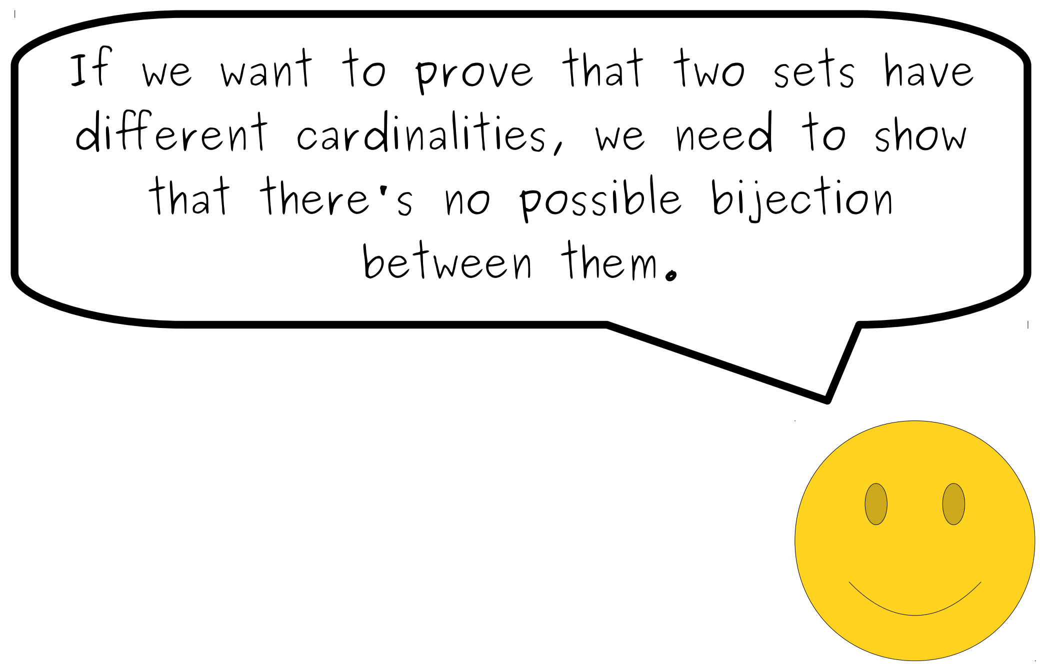

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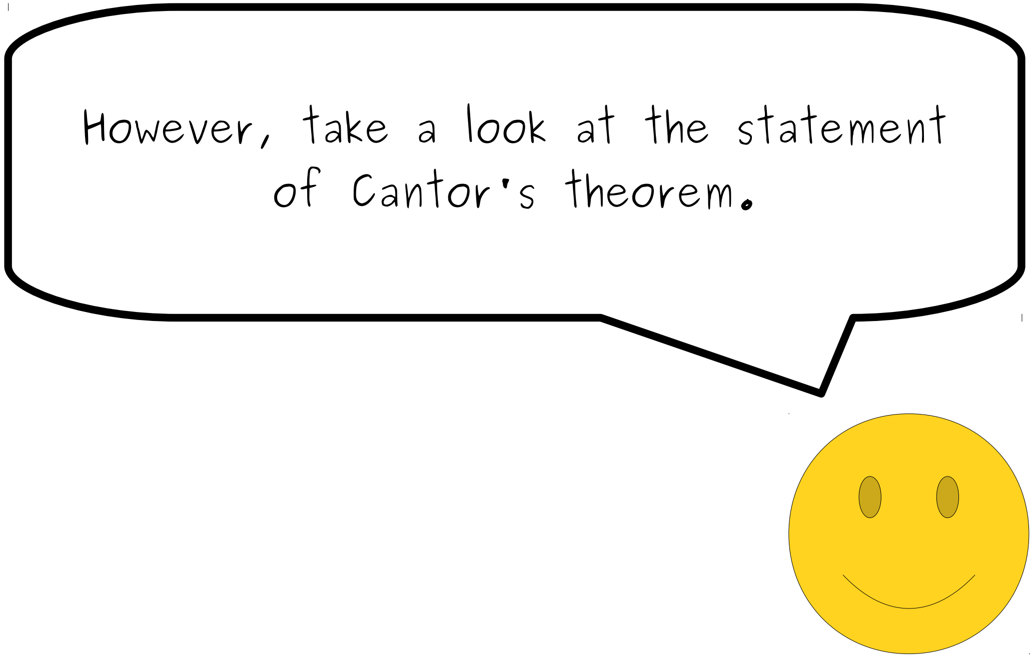

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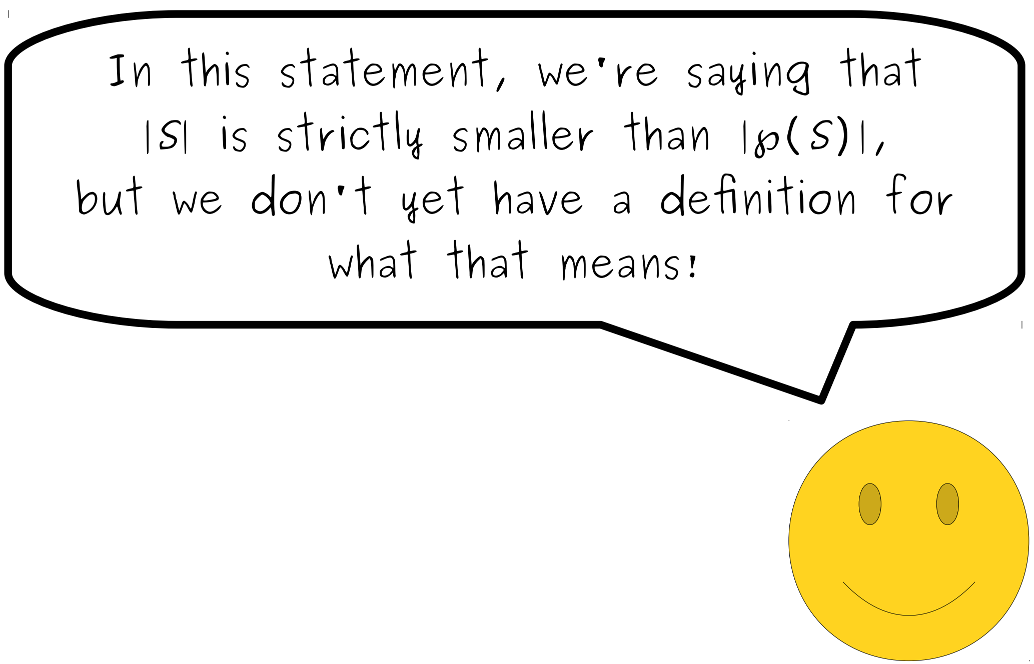

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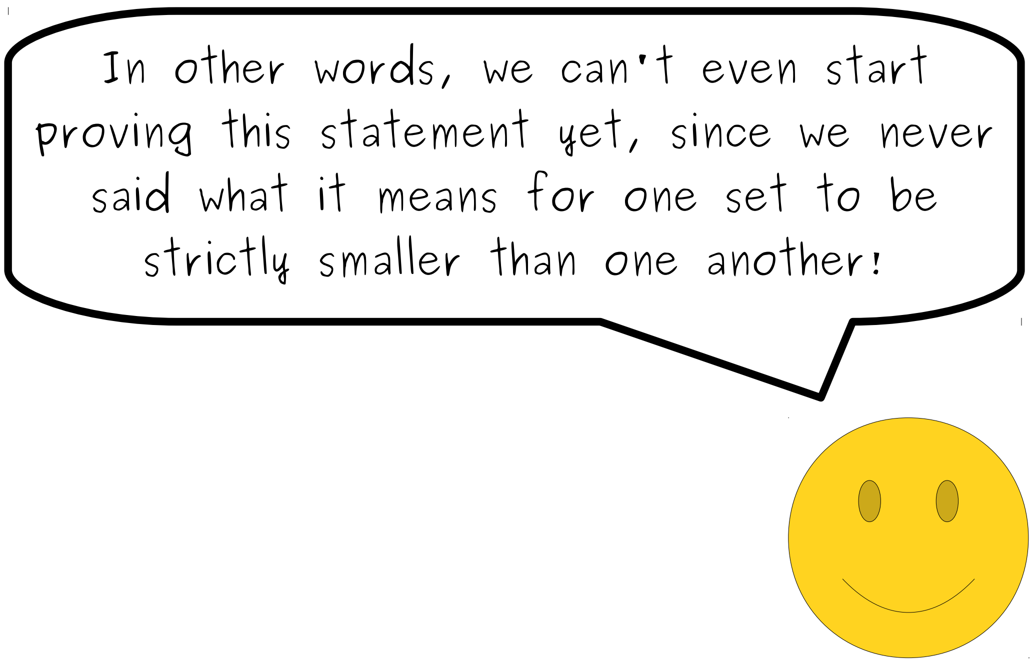

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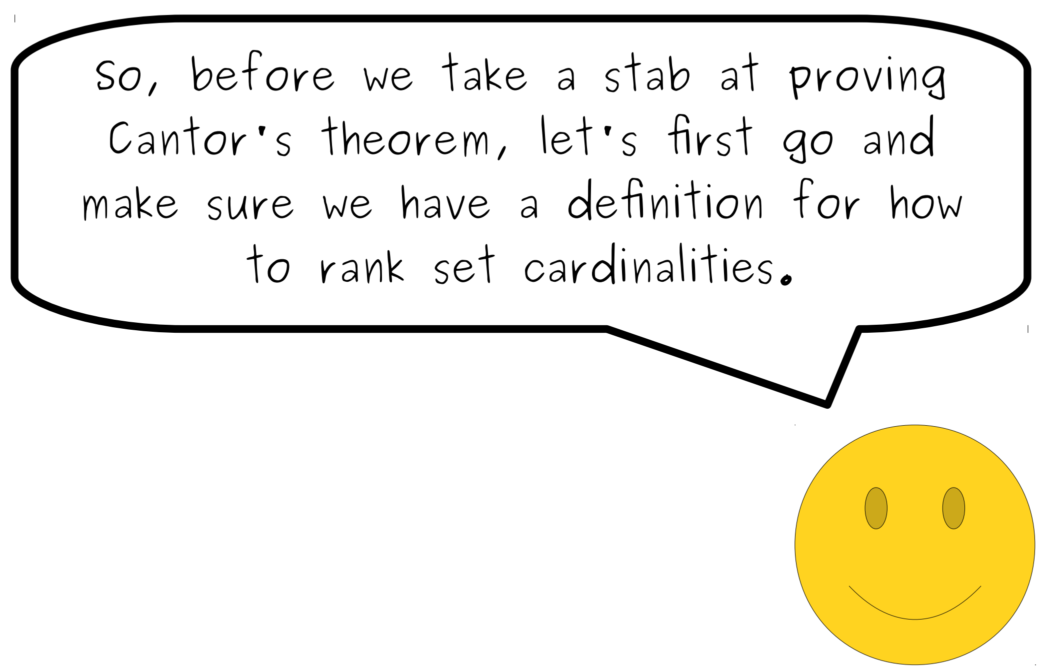
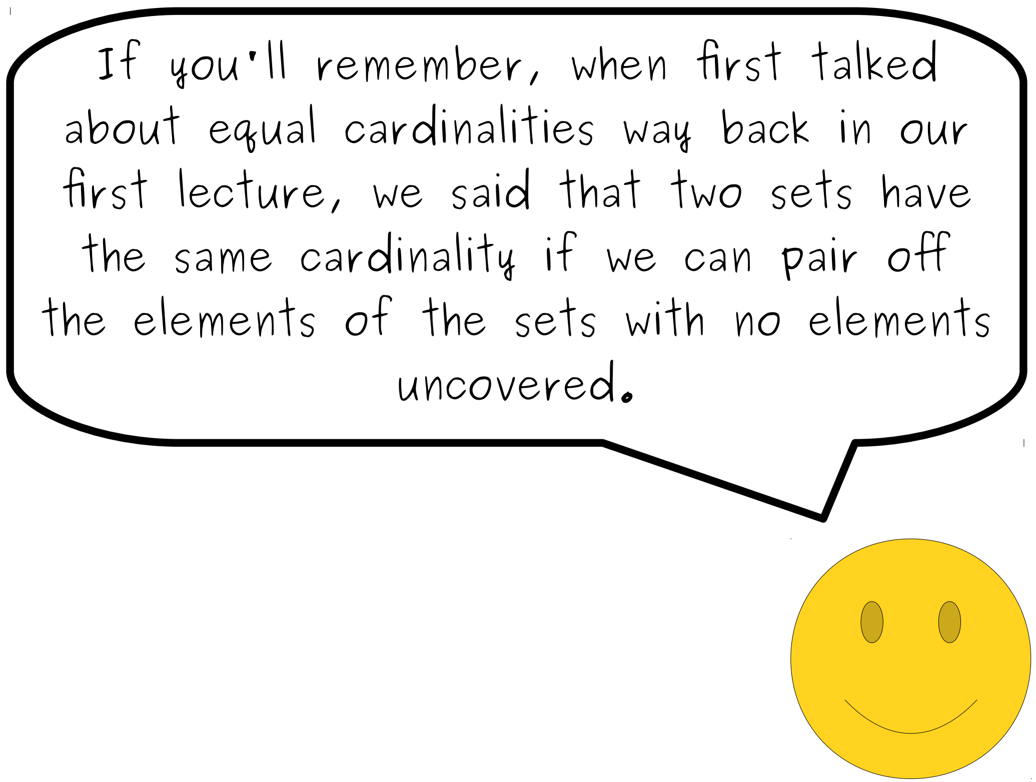
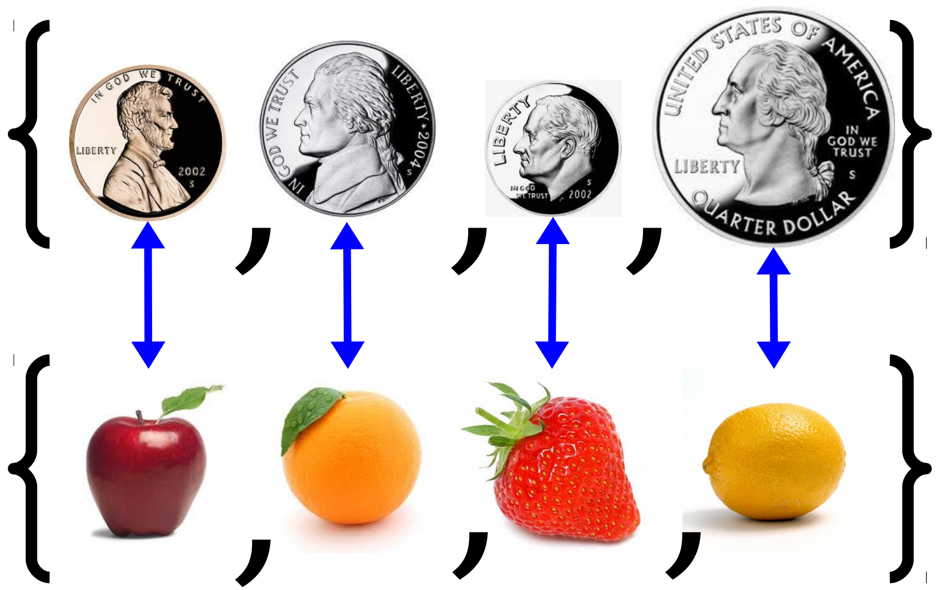
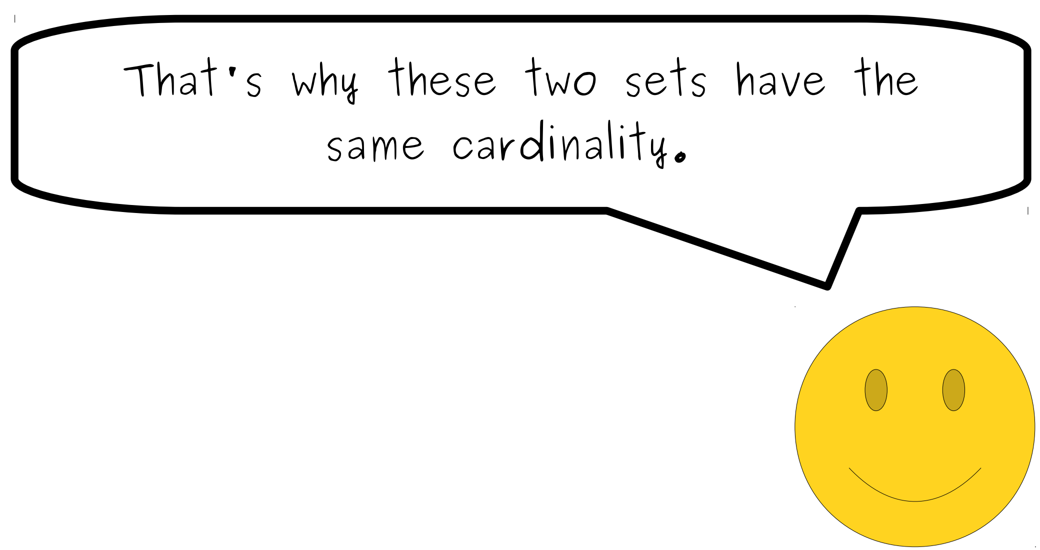
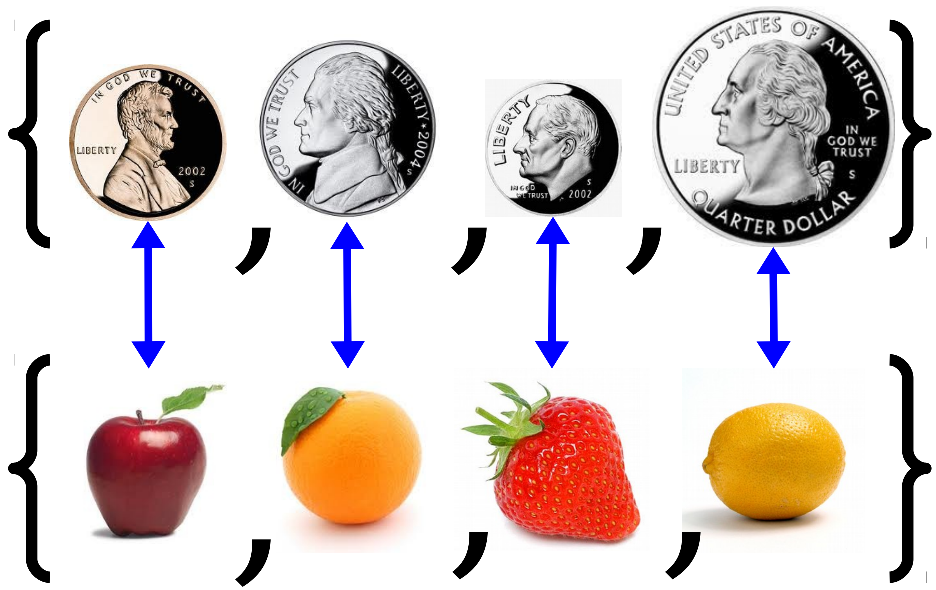


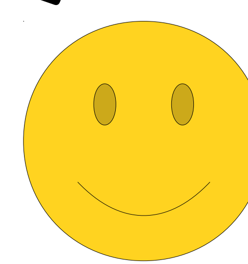

If you'll remember, when first talked about equal cardinalities way back in our first lecture, we said that two sets have the same cardinality if we can pair off the elements of the sets with no elements uncovered.




That's why these two sets have the same cardinality.

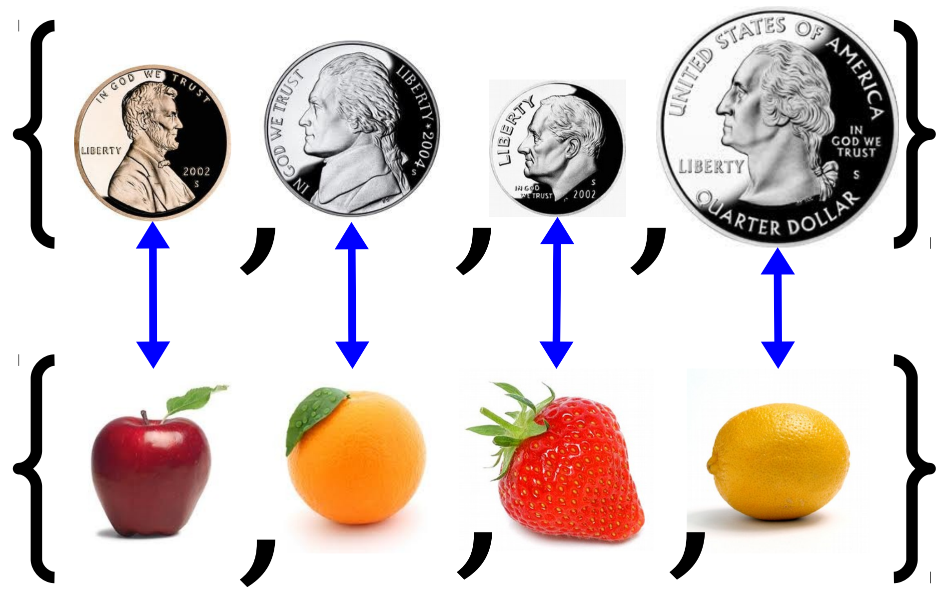


Once we had the language of bijections, we were able to give a more formal definition of what "pairing things off" means.

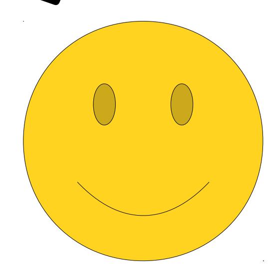


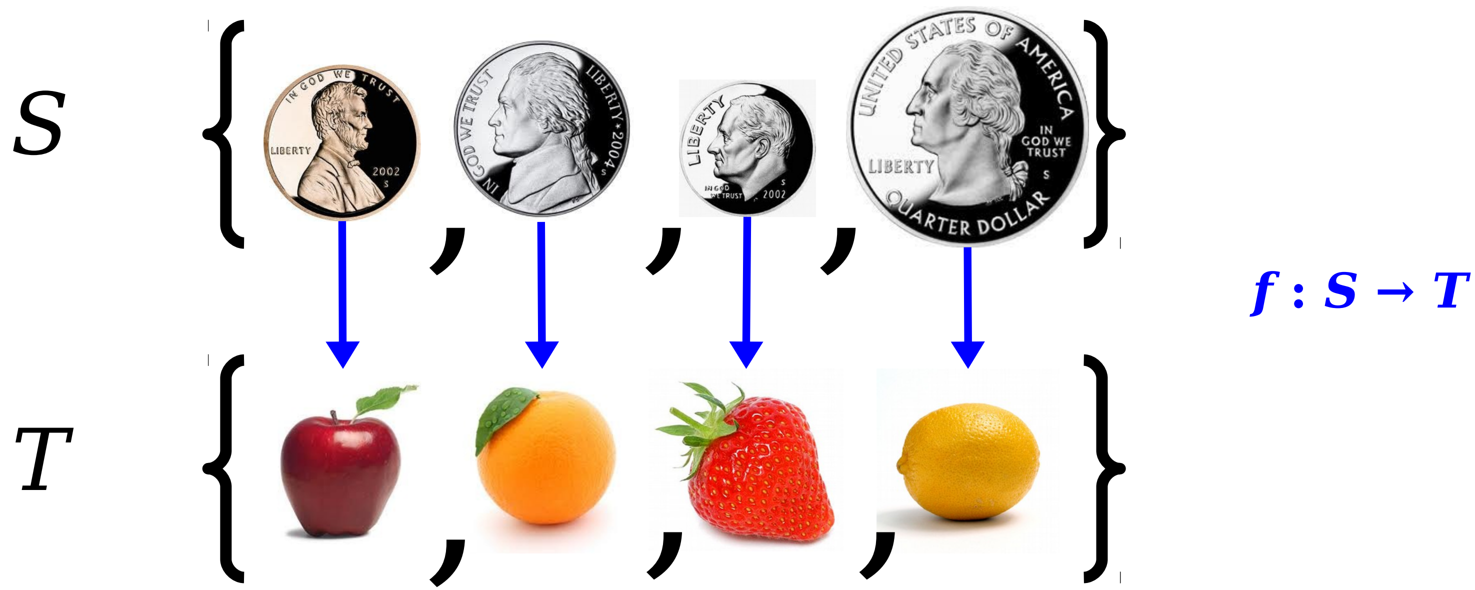
S

T

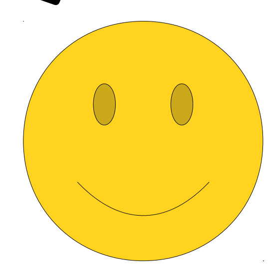


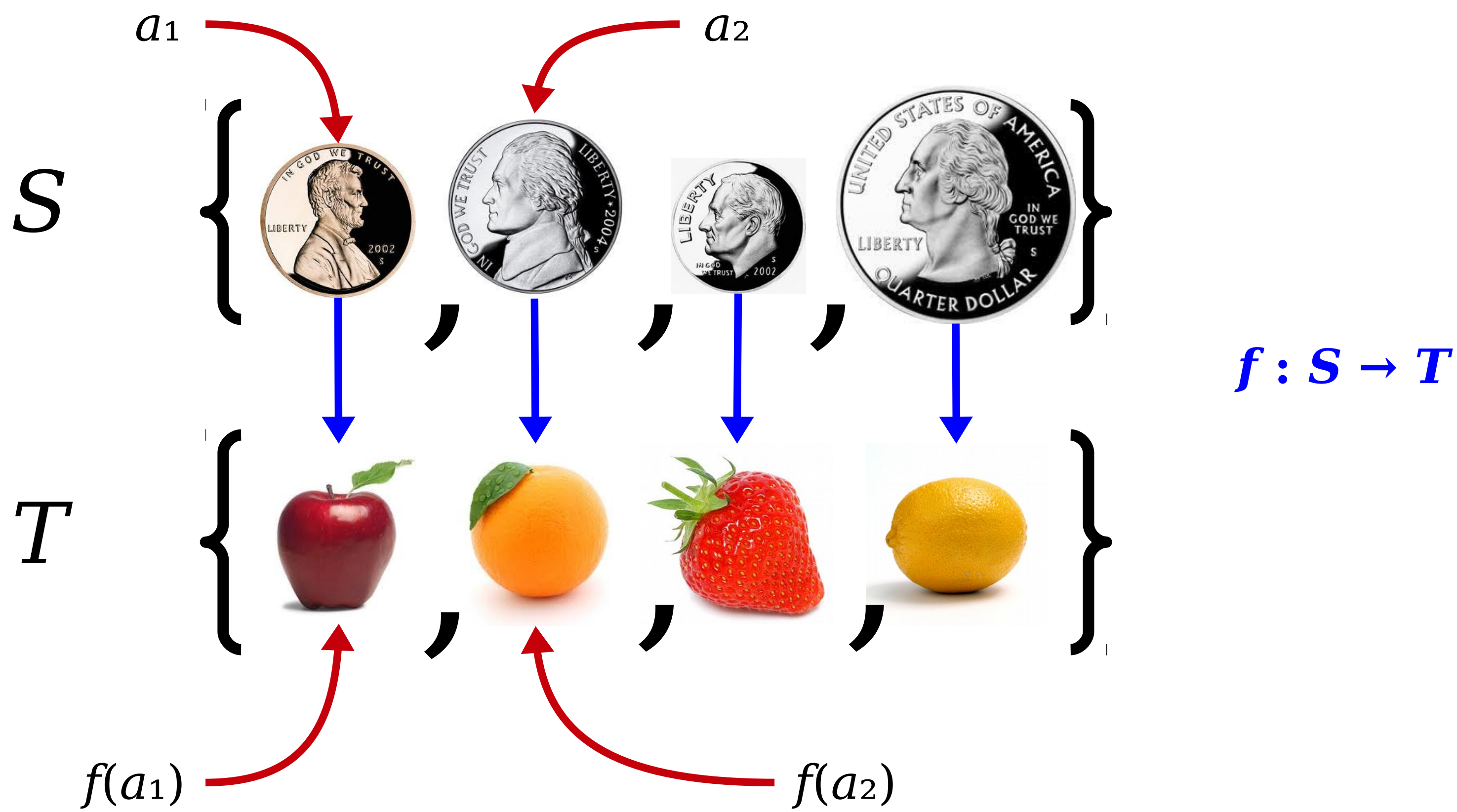
Specifically, if we label these two sets S and T ...



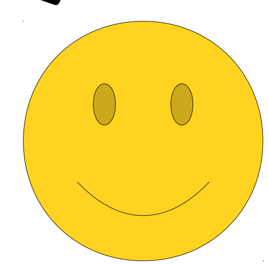


We can think of the pairing as a function from S to T.



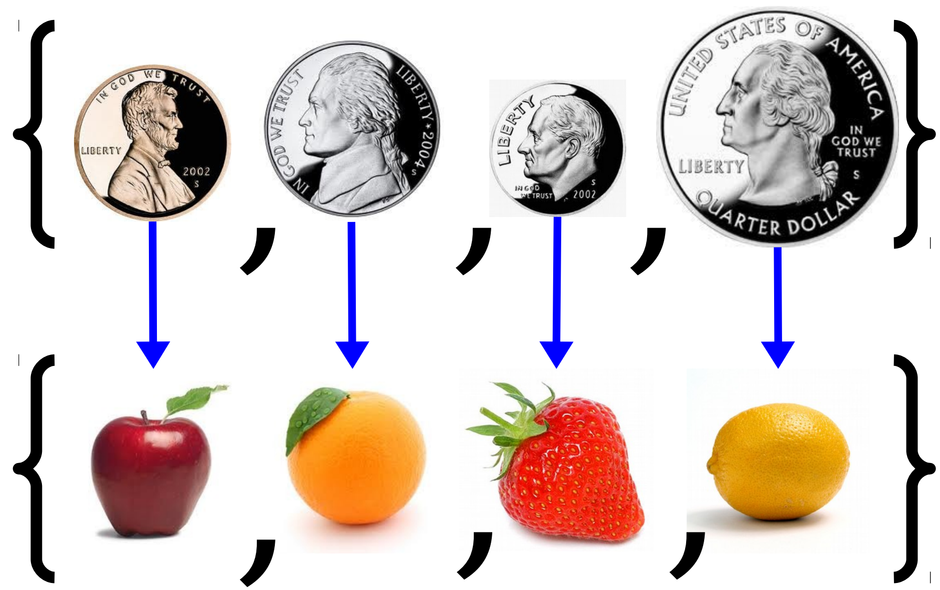


This function will be injective – any two different inputs produce different outputs.



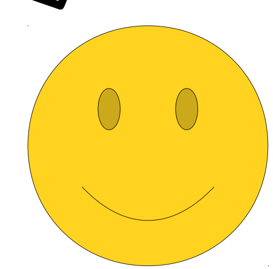
S

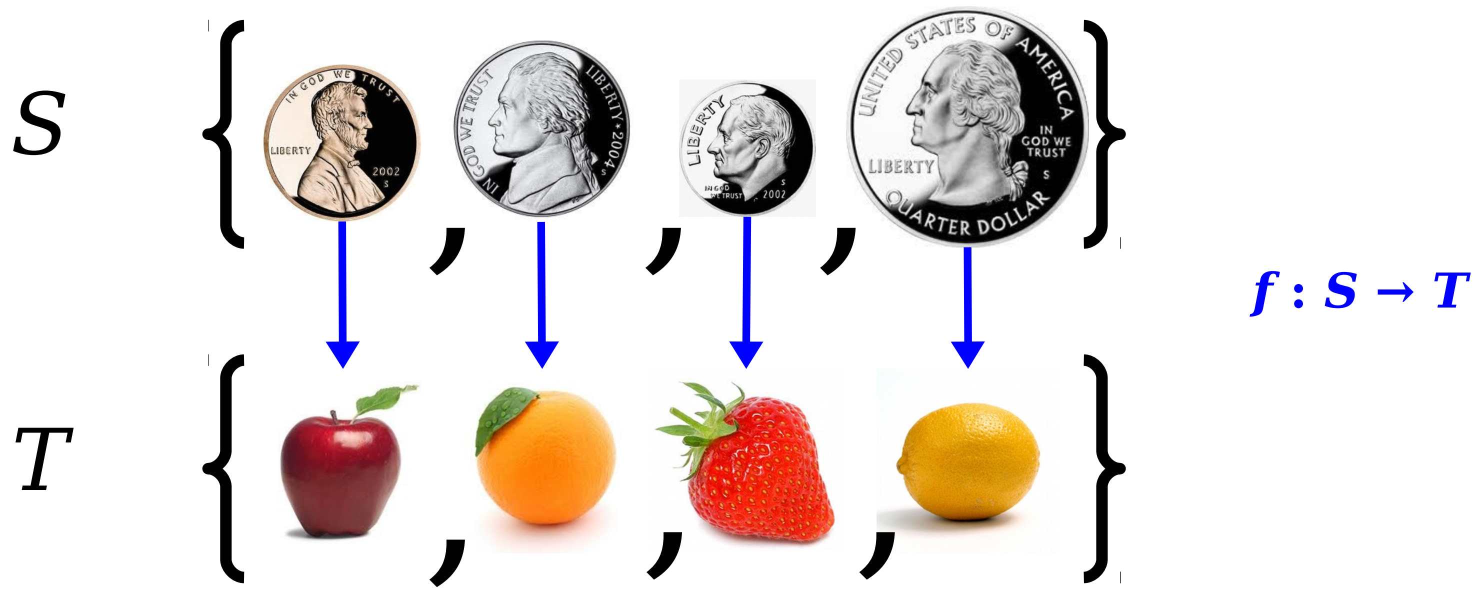
T



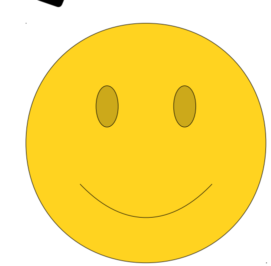
$f : S \to T$

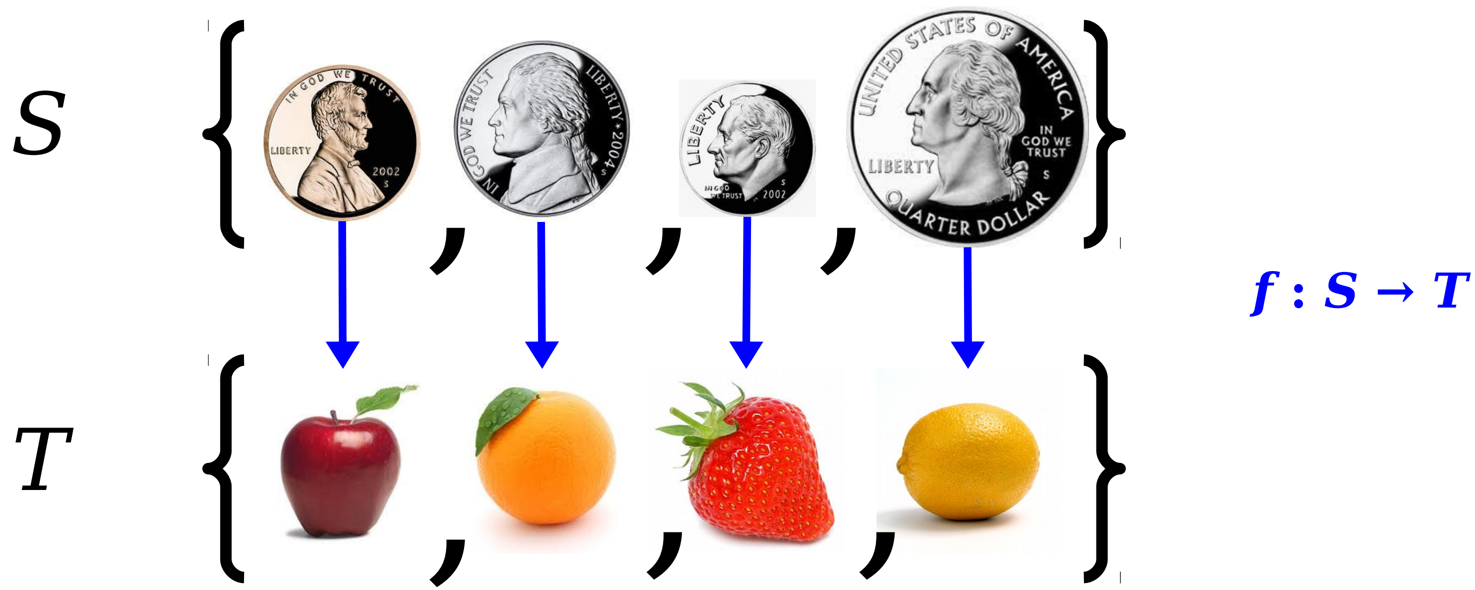
It's also surjective, since any element of the codomain (here, T) will be paired with something in S.



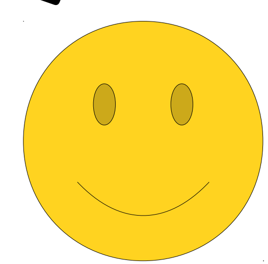


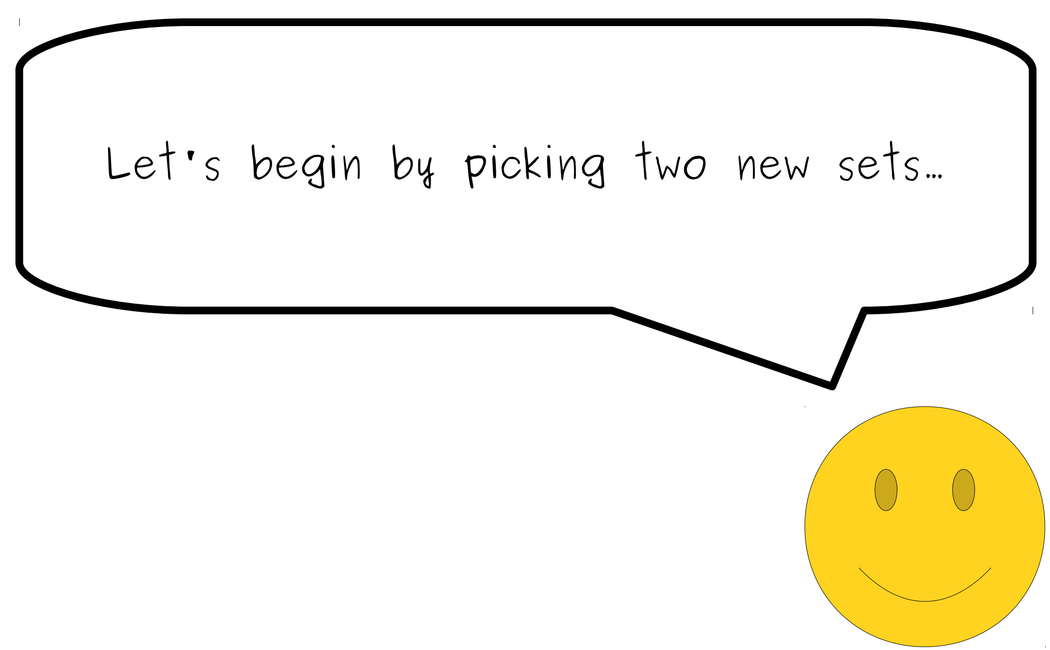
That's where we get the bijection-based definition of equal set cardinality from.



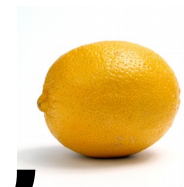


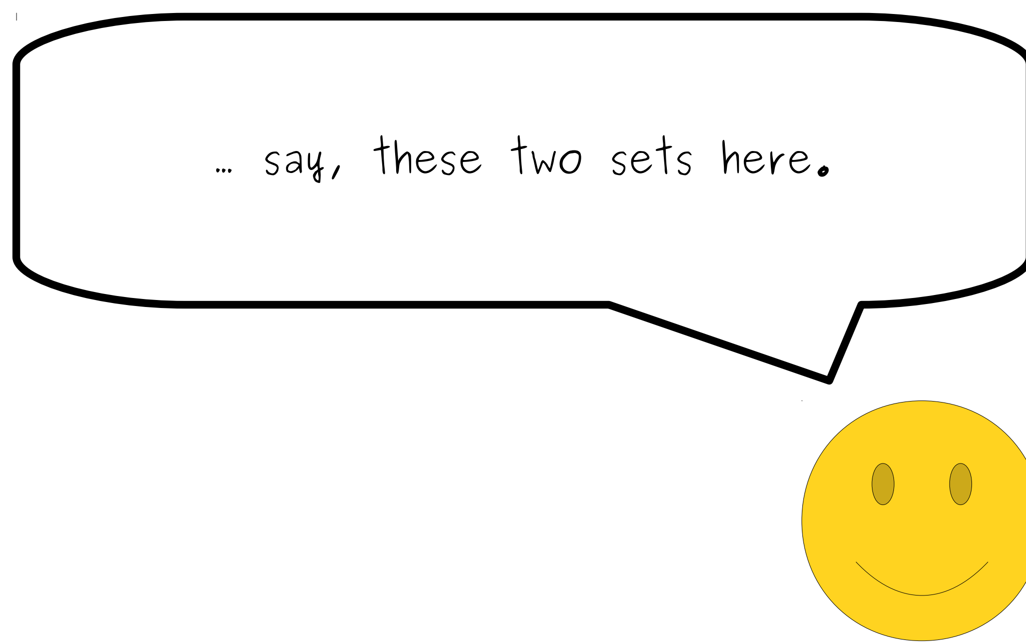
Could we develop some sort of analogous definition for what it means for one set to be "at least as big" as another?



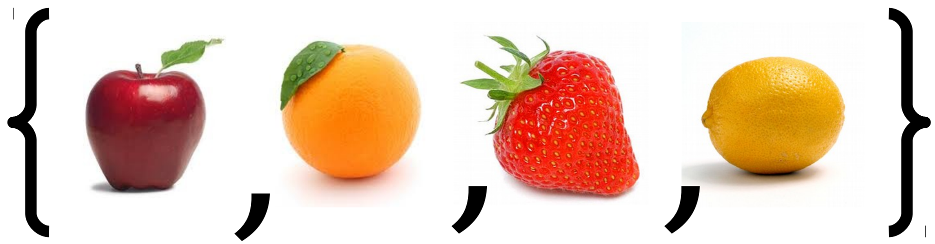

Let's begin by picking two new sets...

S $\{$  ,  ,  $\}$

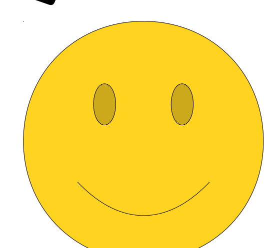
T $\{$  ,  ,  ,  $\}$

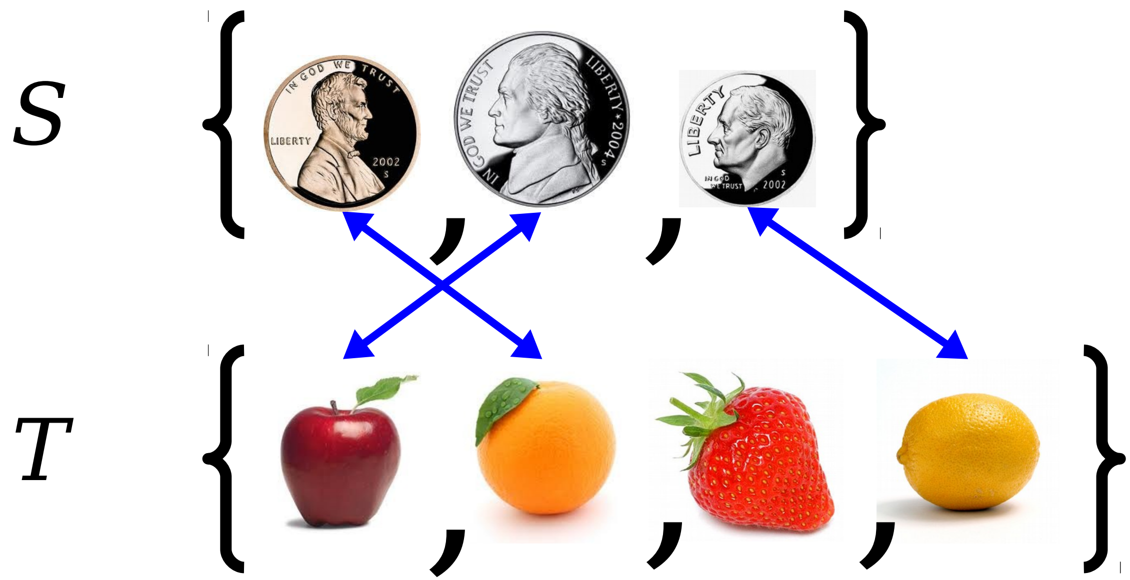


S 

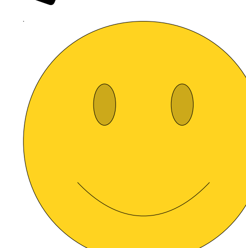
T 

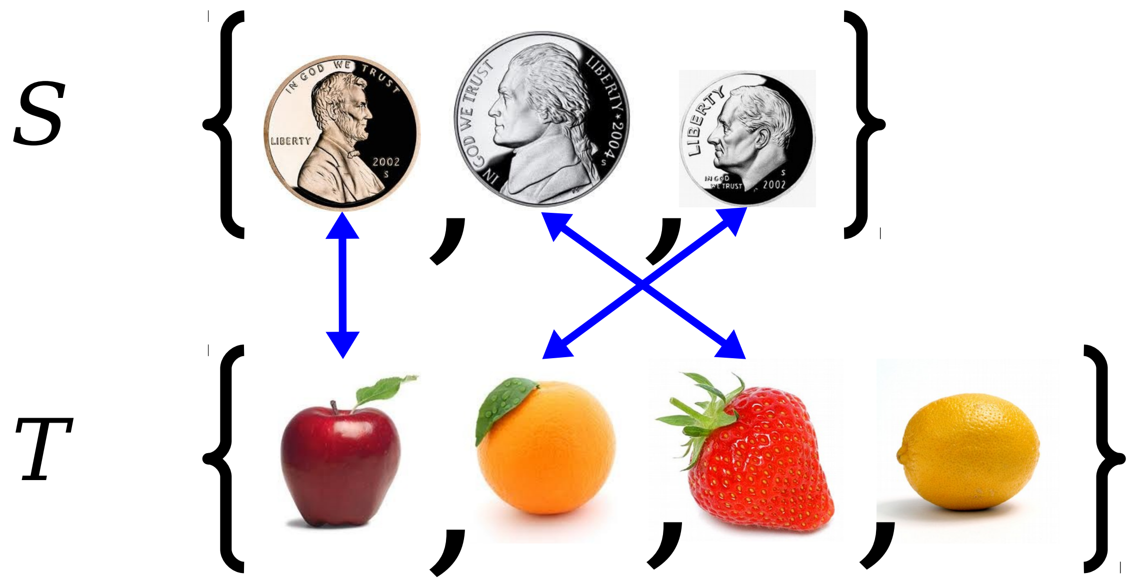
If we try pairing off the elements of S and the elements of T...



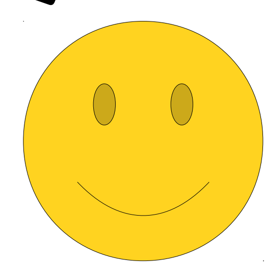


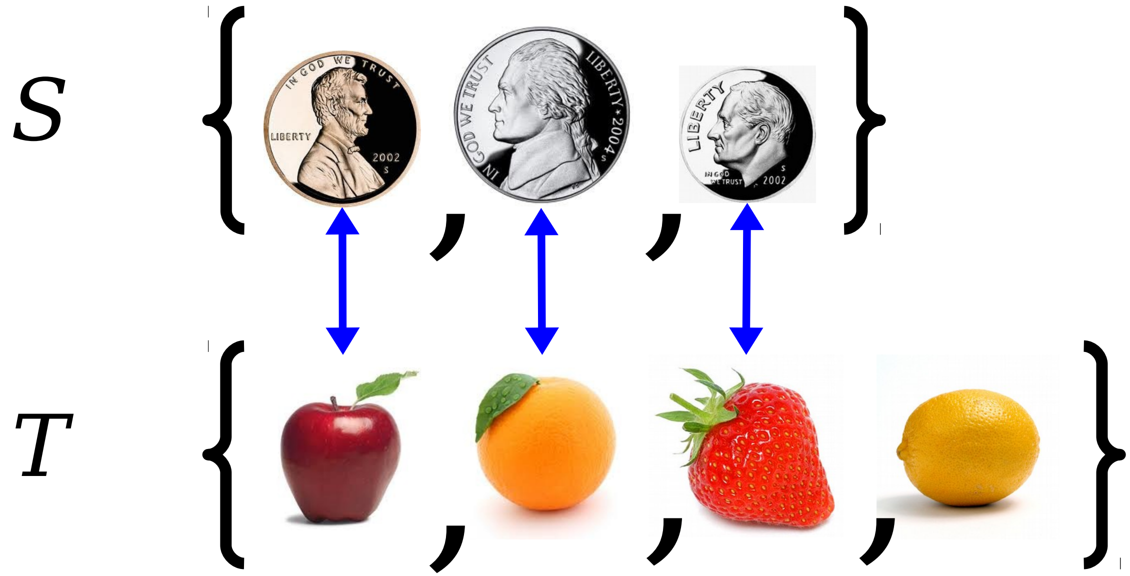
…we can see that there's some way to do it that uses up everything from S.

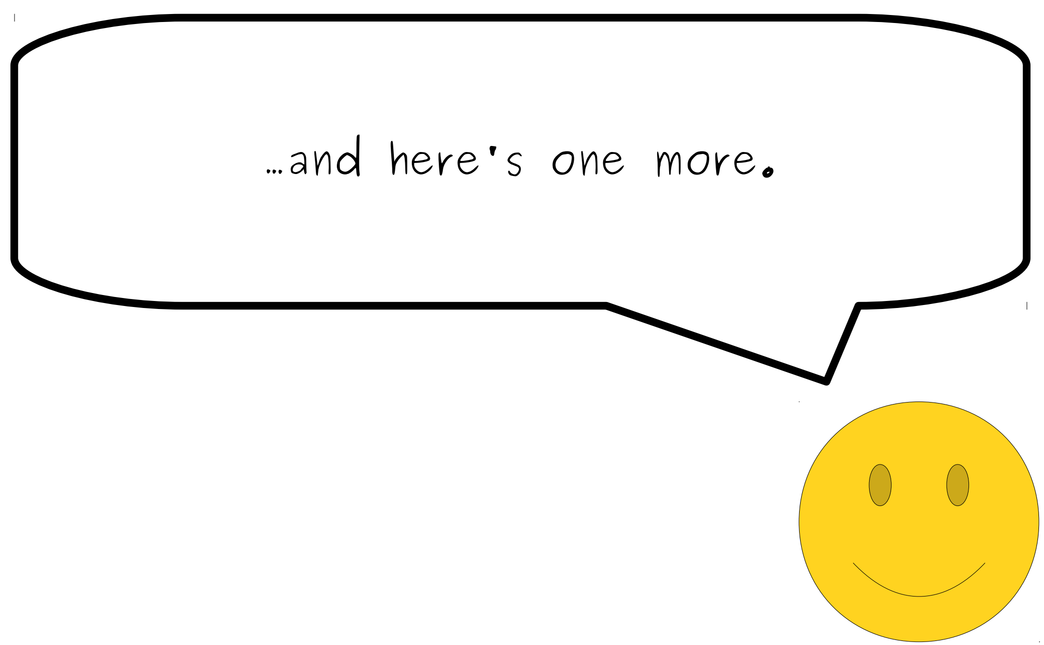


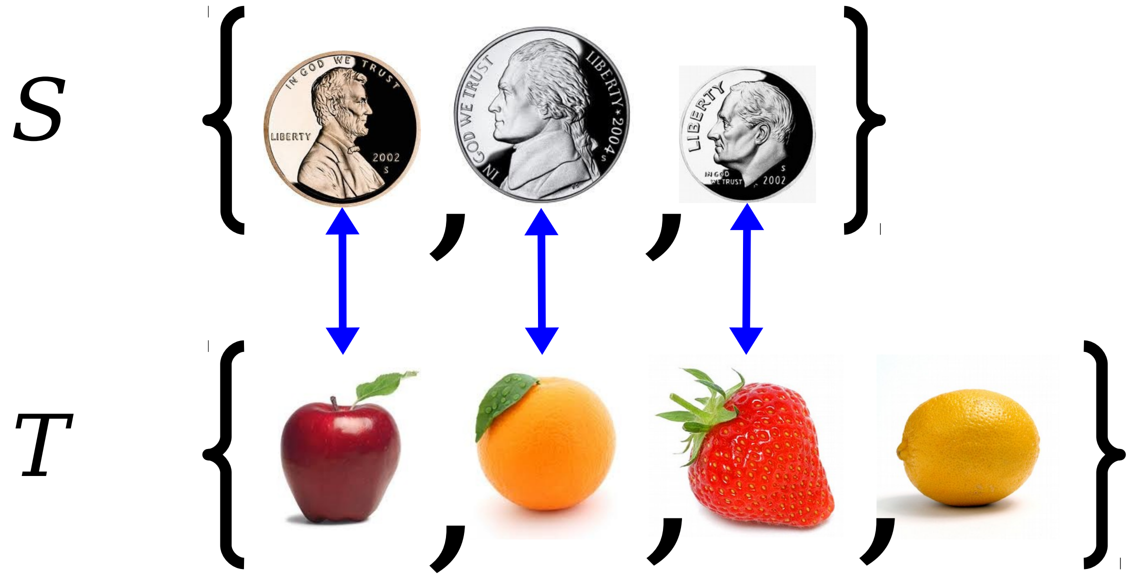


There's actually a bunch of ways to do this. Here's a different one...

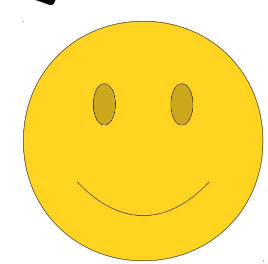



S
T


...and here's one more.

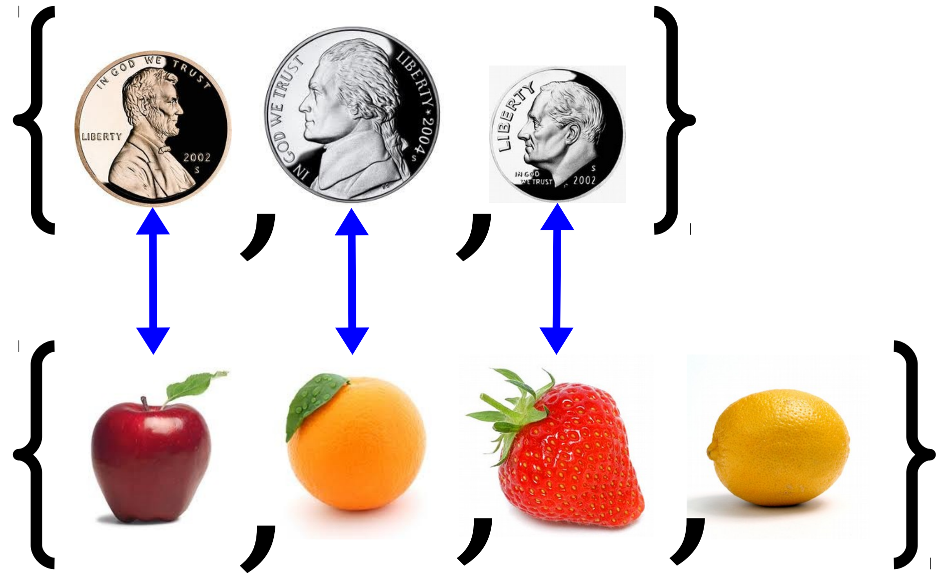


This gives us a nice intuition for what it means for one set to have at least as many elements as the other.

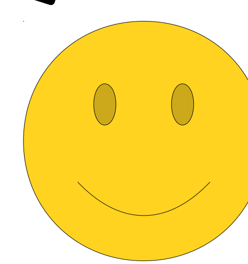


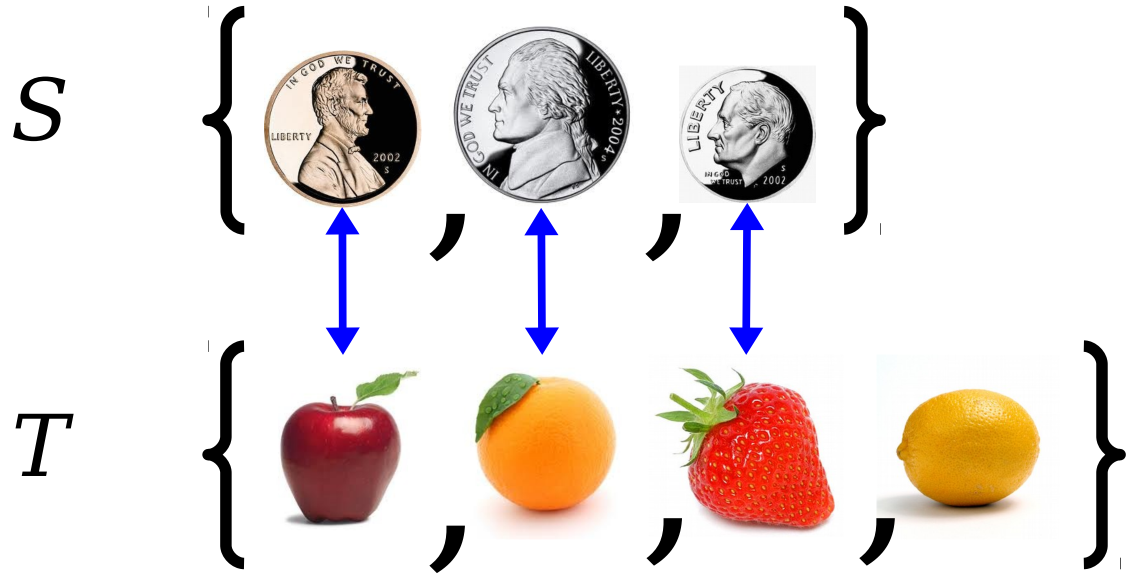
S

T

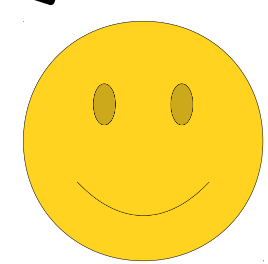


Specifically, we'll say that T is at least as big as S if there's a way to pair the elements that uses up everything in S, even if it doesn't use up everything in T.



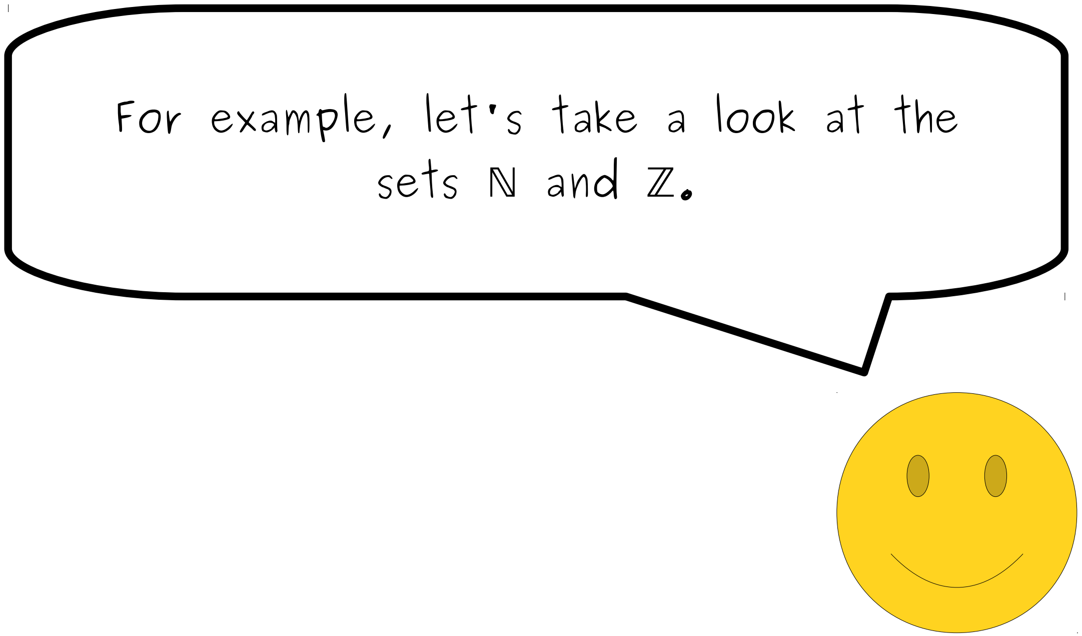


Before moving on, let's make sure that this definition works in some other cases. If it doesn't, then chances are it's not a very good definition!

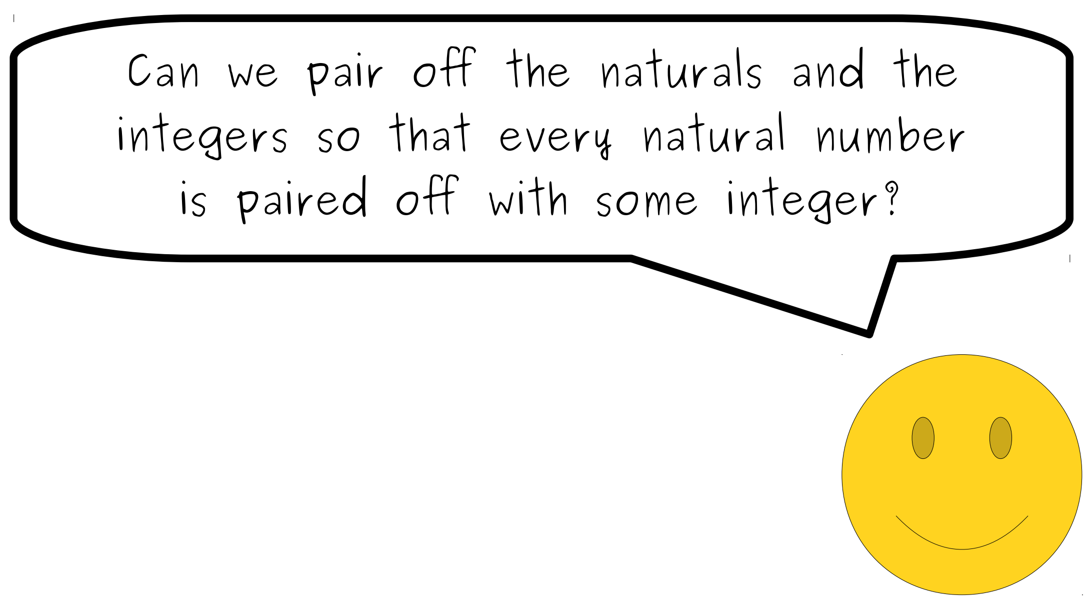


$\mathbb{N}$ $\quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots$

$\mathbb{Z}$ $\quad \ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots$

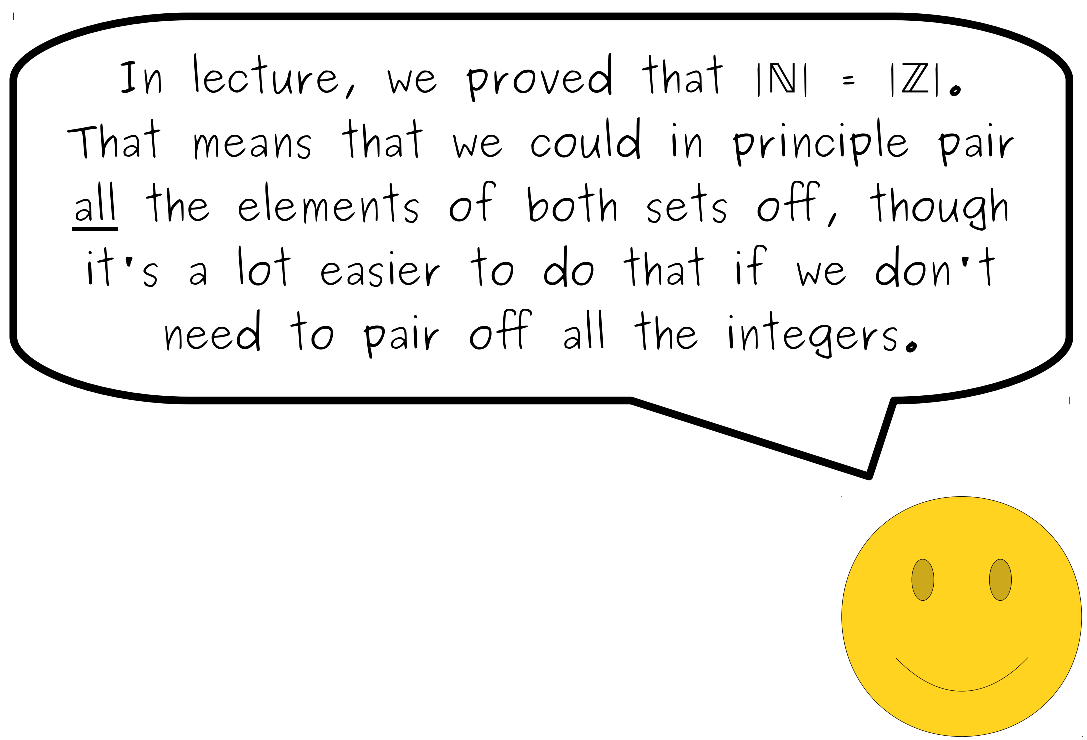


$\mathbb{N}$					0	1	2	3	…
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	…

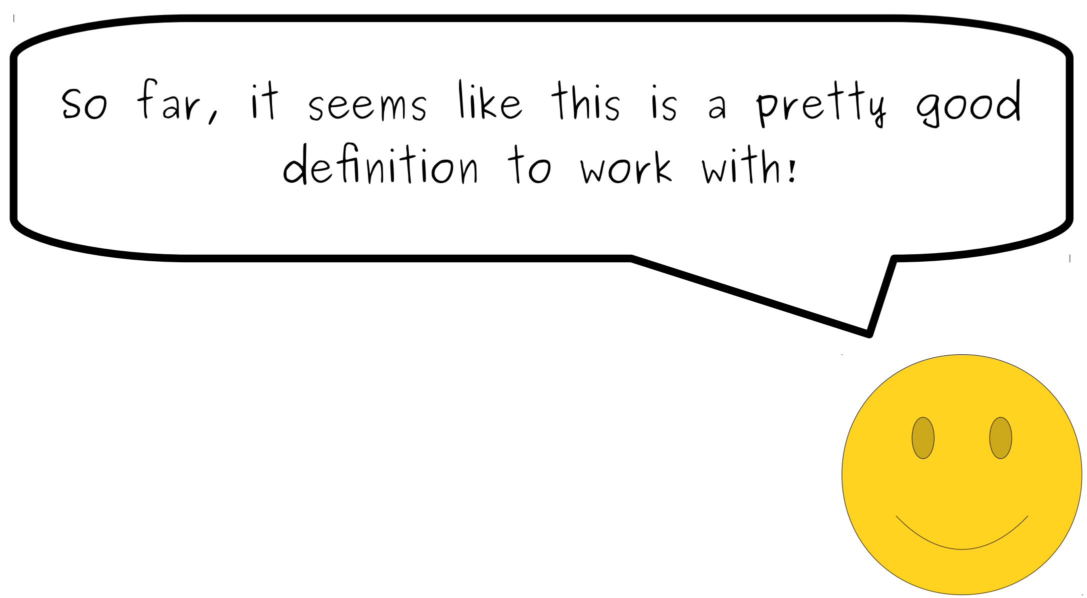


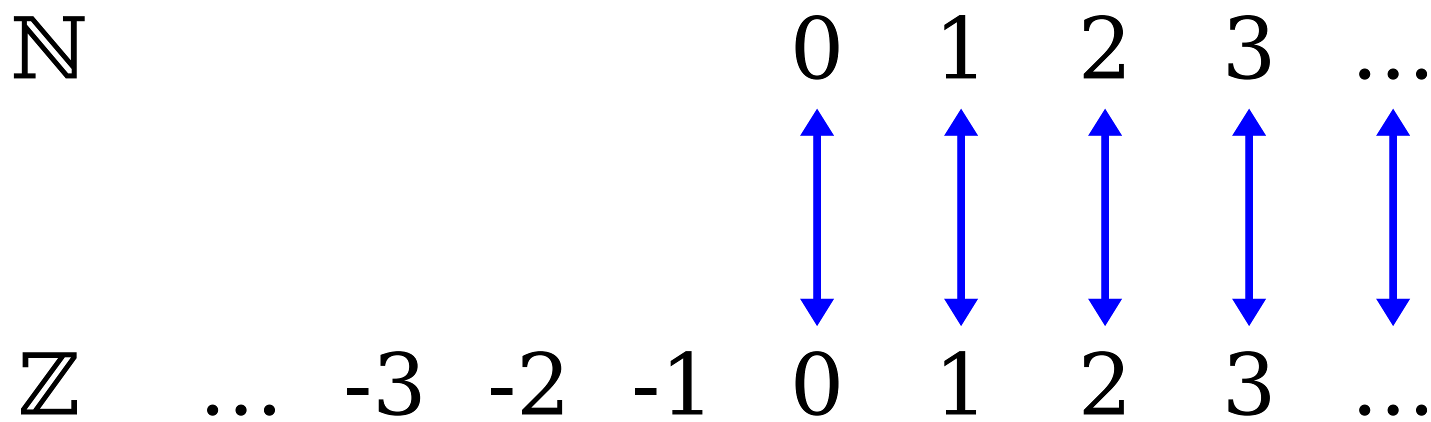
$\mathbb{N}$					0	1	2	3	…
					$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	…

$\mathbb{N}$				0	1	2	3	…
				↕	↕	↕	↕	↕
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	…

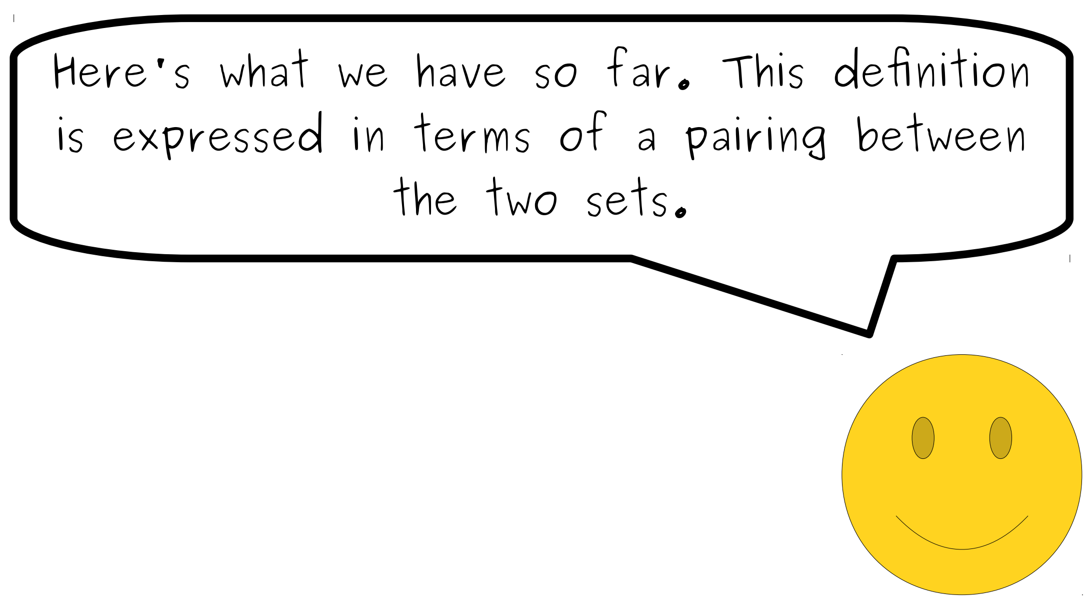


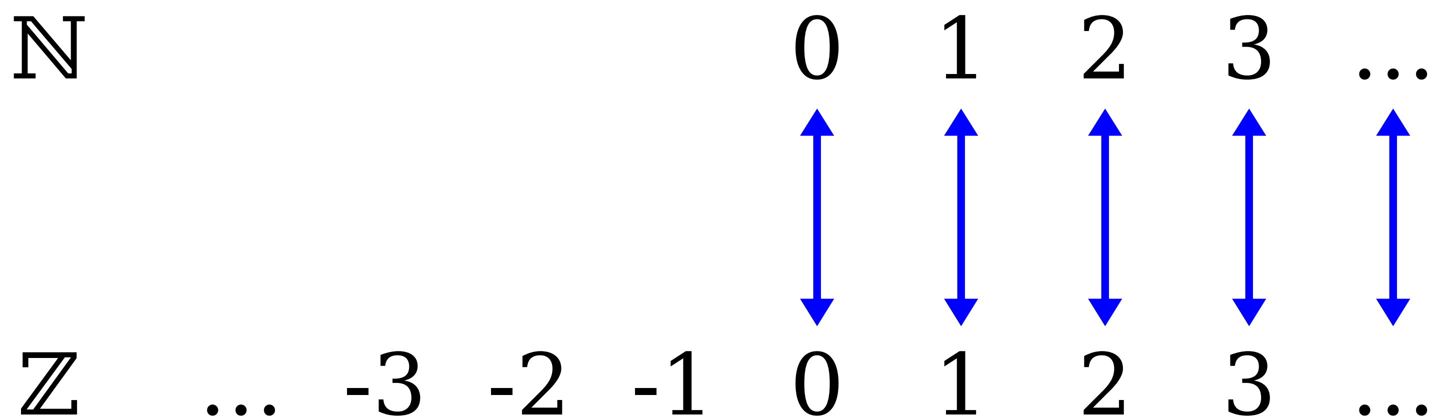
$$\begin{array}{cccccccccc}
\mathbb{N} & & & & & 0 & 1 & 2 & 3 & \dots \\
 & & & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\mathbb{Z} & \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \dots
\end{array}$$





$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.





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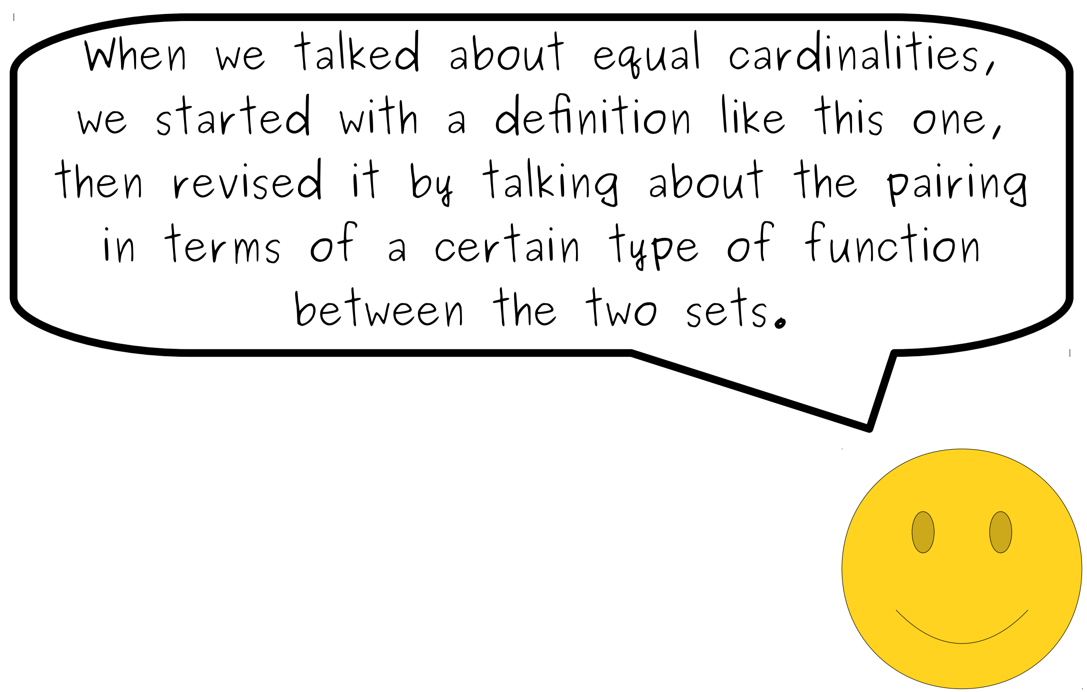
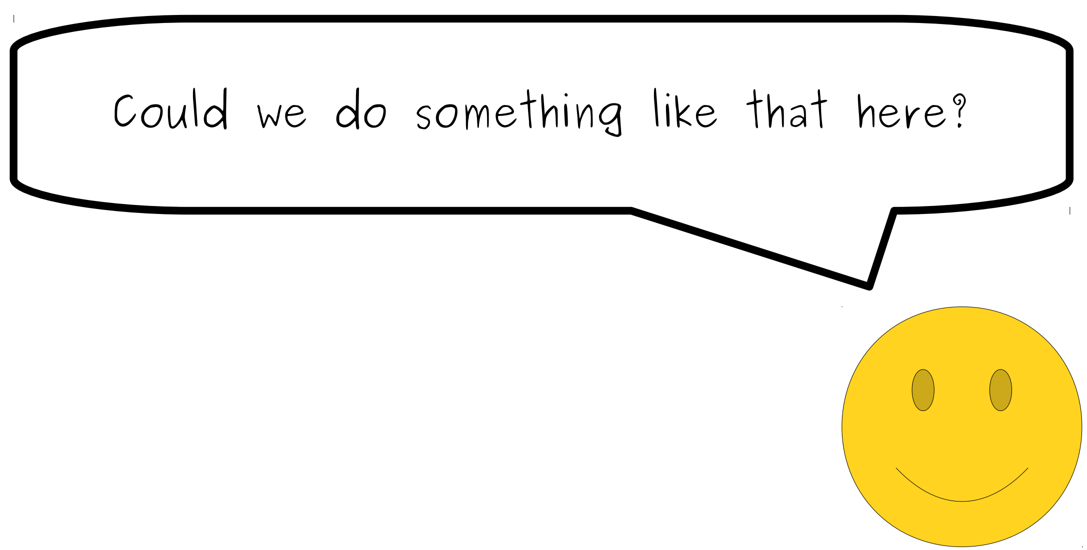


$$\begin{array}{lccccccccc} \mathbb{N} & & & & & 0 & 1 & 2 & 3 & \ldots \\ & & & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathbb{Z} & \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \end{array}$$

$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.



$\mathbb{N}$					0	1	2	3	…
					$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	…

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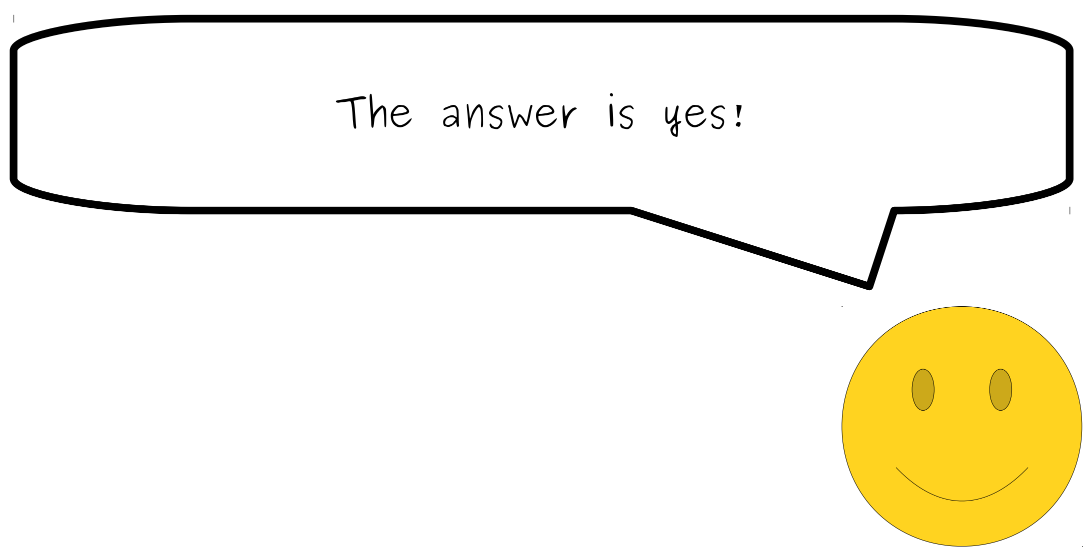

$\mathbb{N}$					0	1	2	3	…
					$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	…

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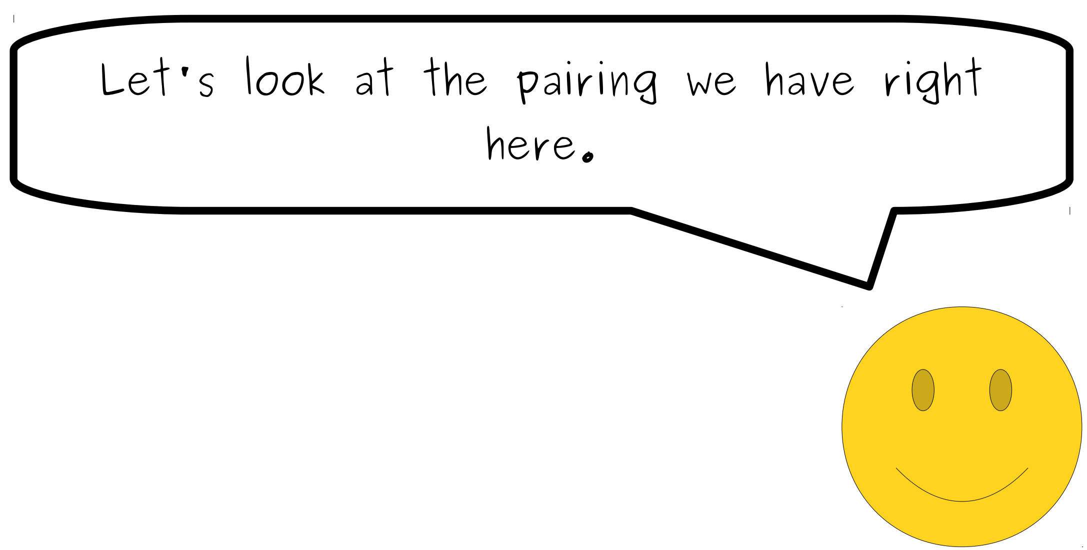
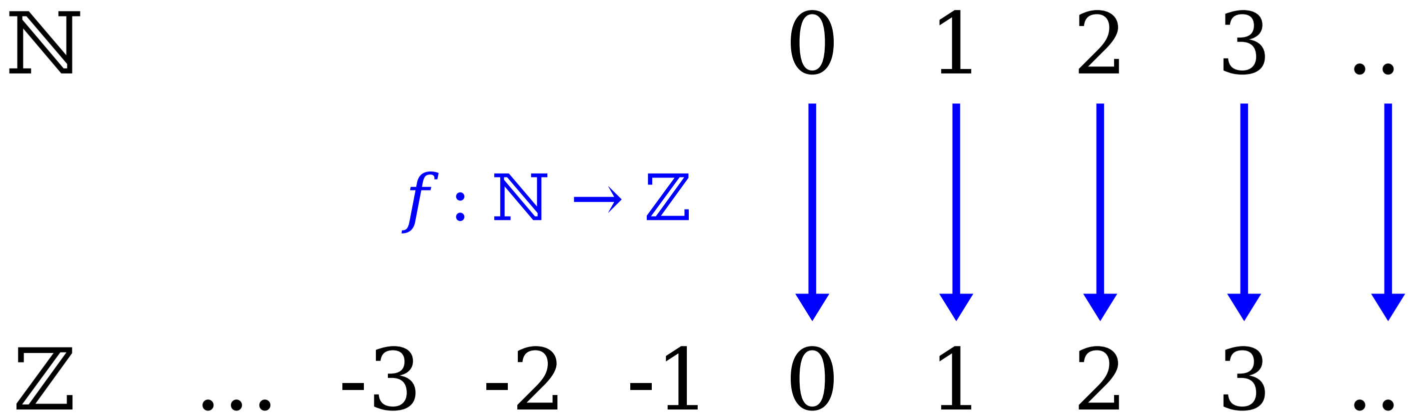
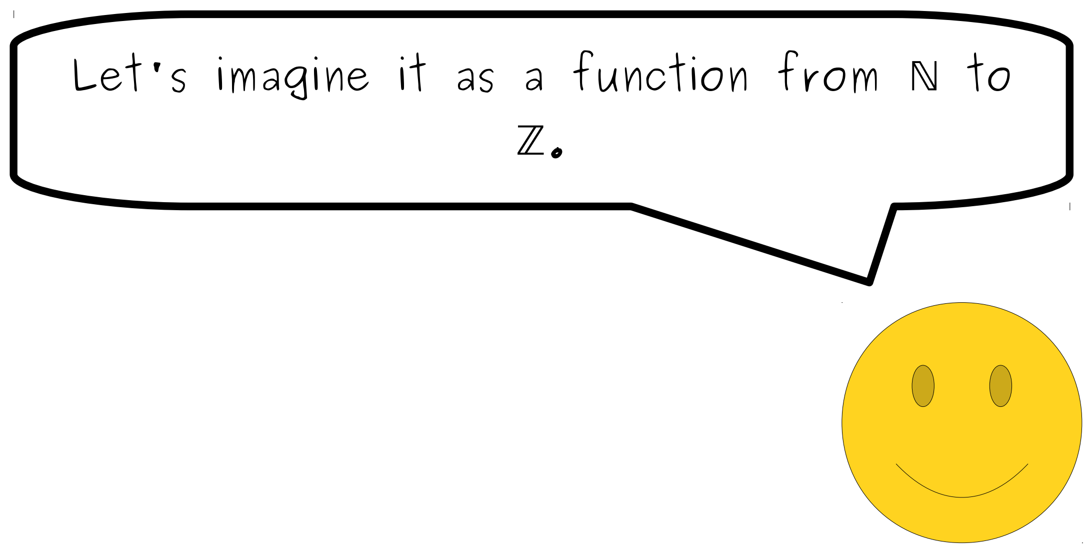




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

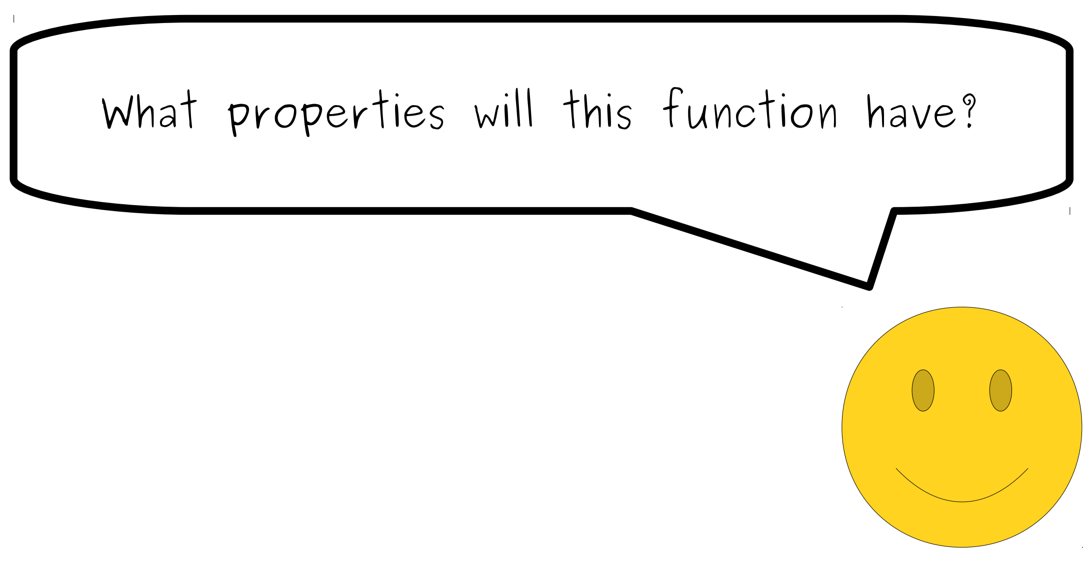


$\mathbb{N}$ 0 1 2 3 …

$f : \mathbb{N} \to \mathbb{Z}$

$\mathbb{Z}$ … -3 -2 -1 0 1 2 3 …

$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.



$\mathbb{N}$					0	1	2	3	…
				$f : \mathbb{N} \to \mathbb{Z}$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathbb{Z}$	…	-3	-2	-1	0	1	2	3	…

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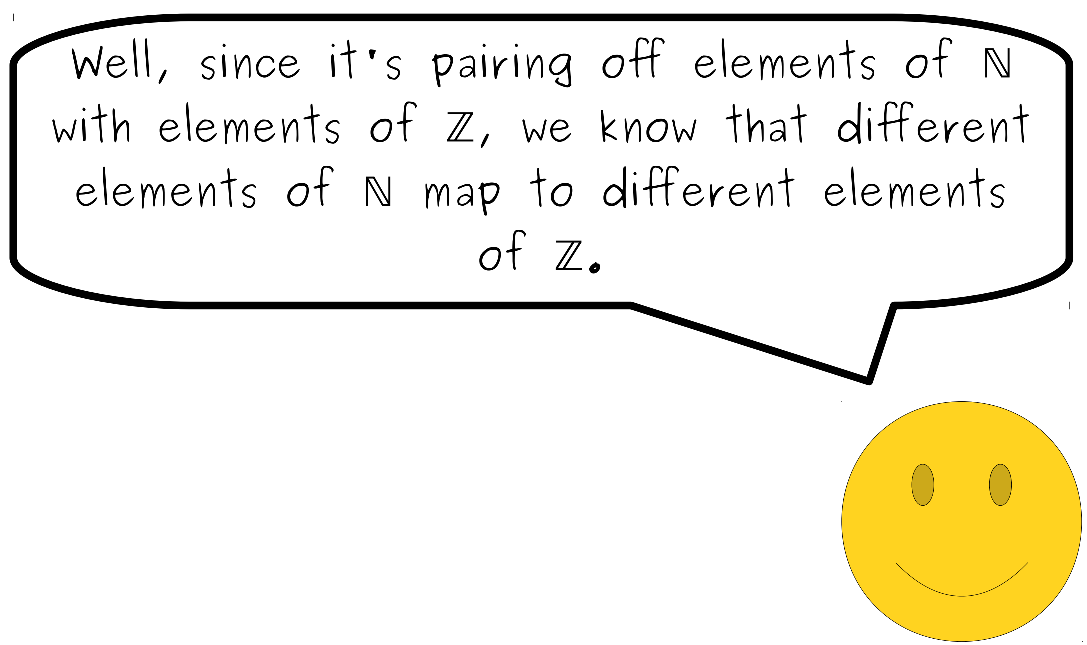
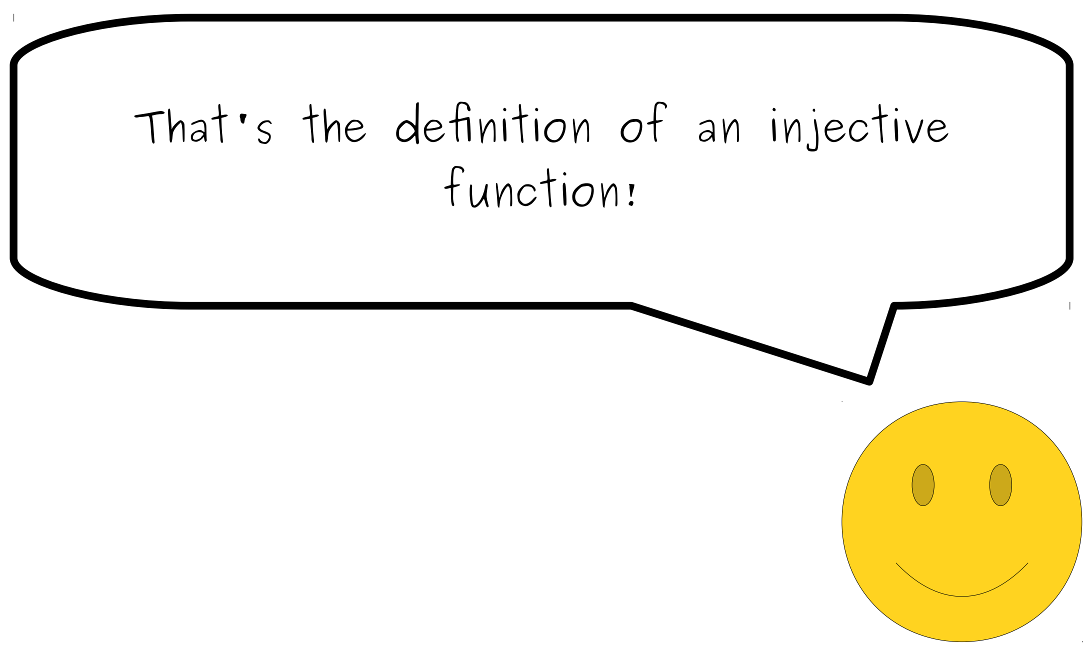


$\mathbb{N}$ 0   1   2   3   …

$f : \mathbb{N} \to \mathbb{Z}$

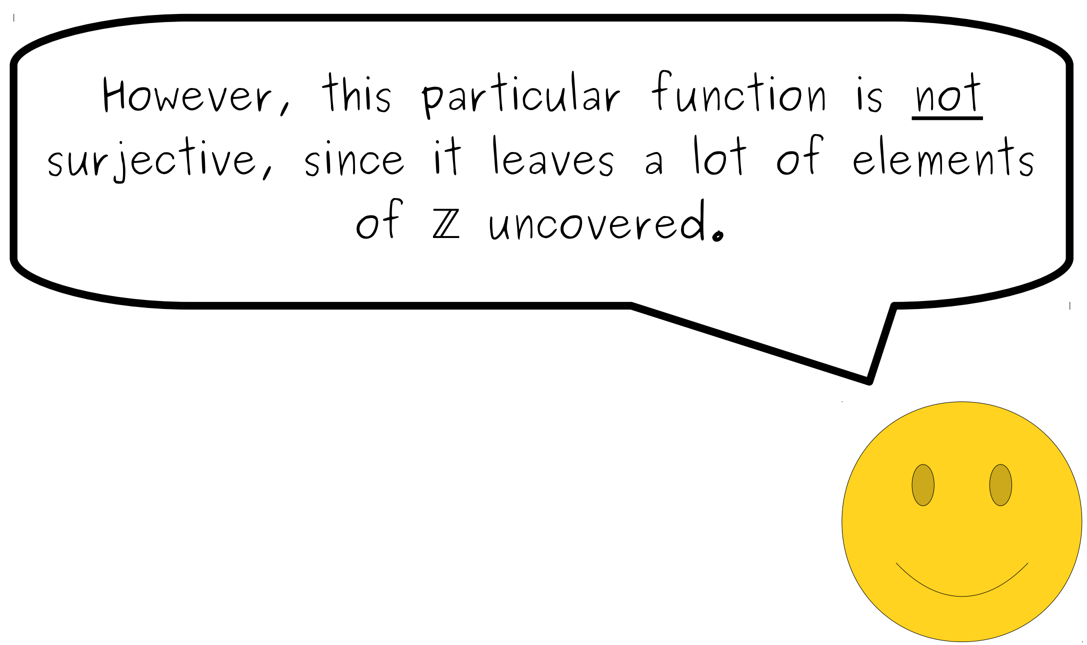
$\mathbb{Z}$ …   -3   -2   -1   0   1   2   3   …

$|S| \leq |T|$ if   the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.



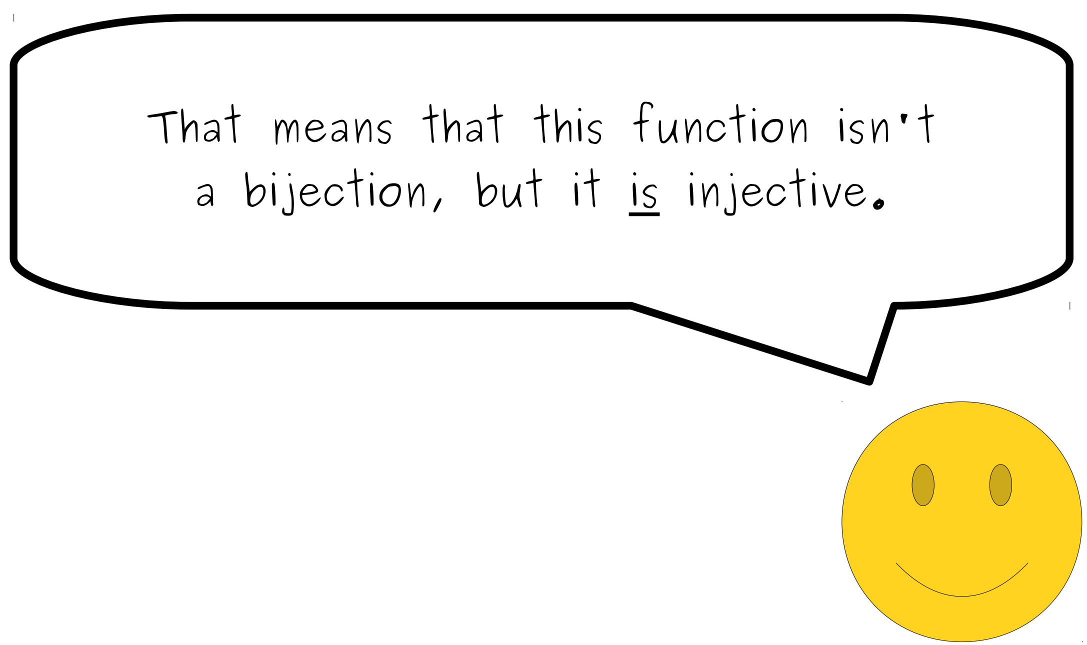
$$
\begin{array}{lcccccccccc}
\mathbb{N} & & & & & 0 & 1 & 2 & 3 & \dots \\
& & & & f : \mathbb{N} \to \mathbb{Z} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Z} & \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \dots
\end{array}
$$

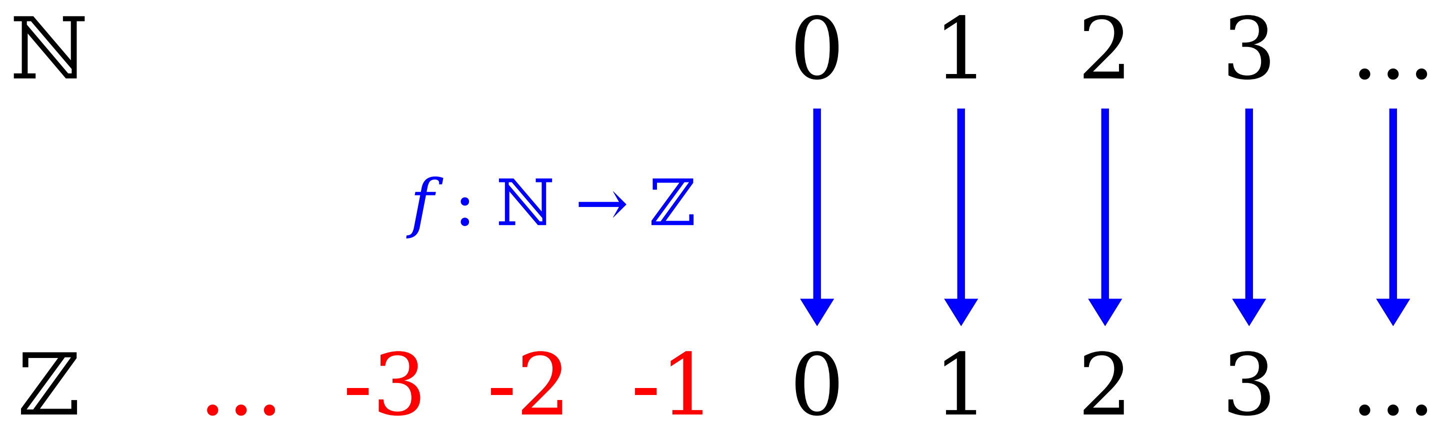
$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.



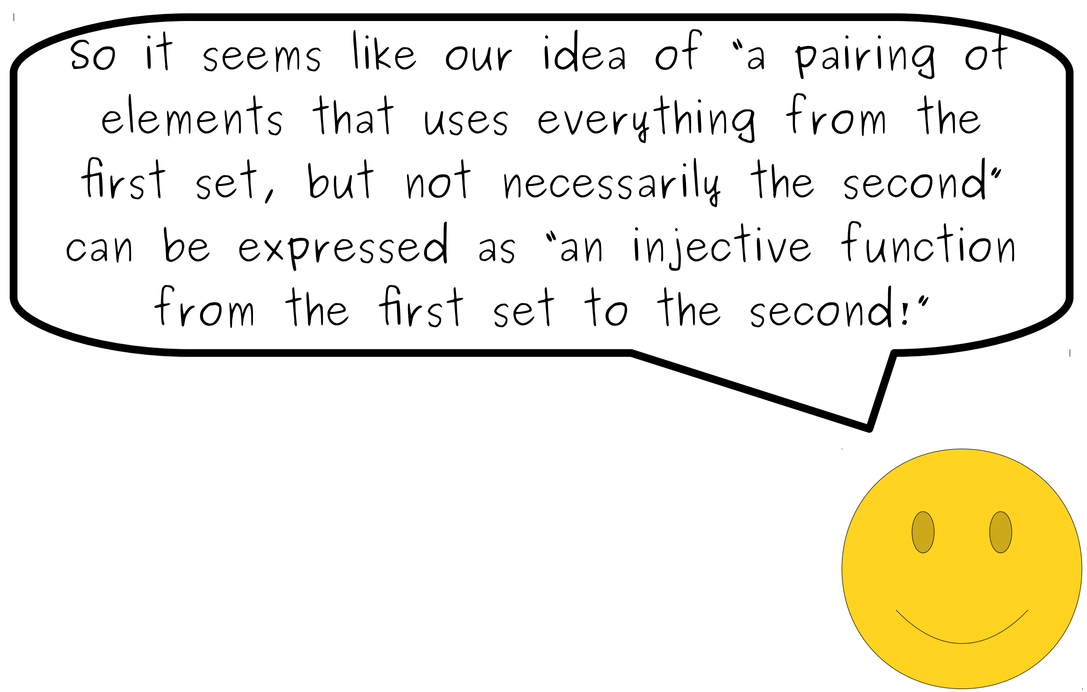
$$
\begin{array}{lcccccccccc}
\mathbb{N} & & & & 0 & 1 & 2 & 3 & \ldots \\
 & & f : \mathbb{N} \to \mathbb{Z} & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Z} & \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \ldots
\end{array}
$$

$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.





$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

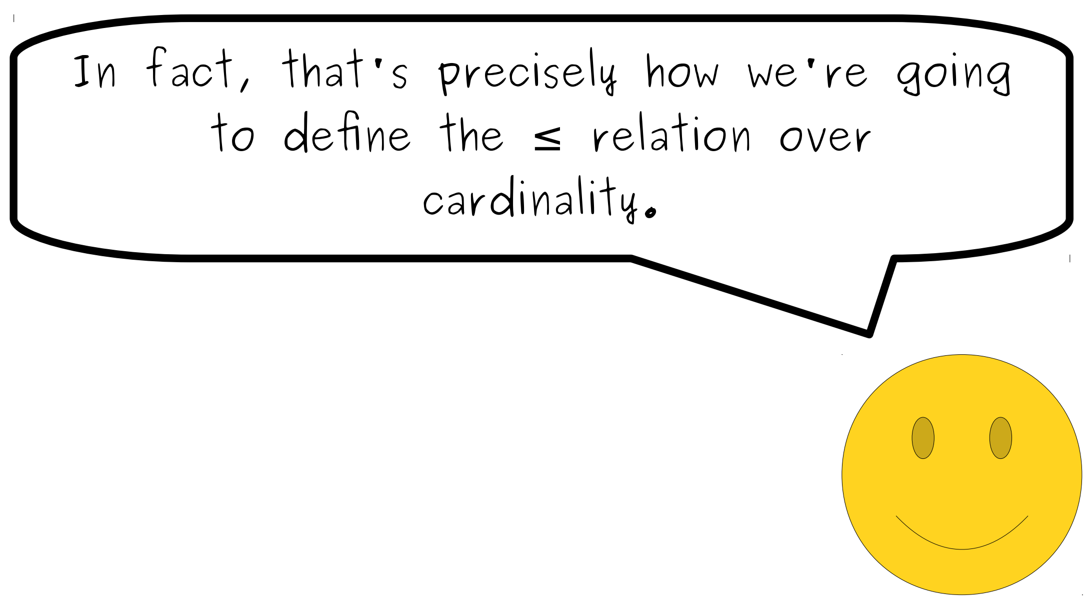


$\mathbb{N}$ 0   1   2   3   ...

$f : \mathbb{N} \to \mathbb{Z}$

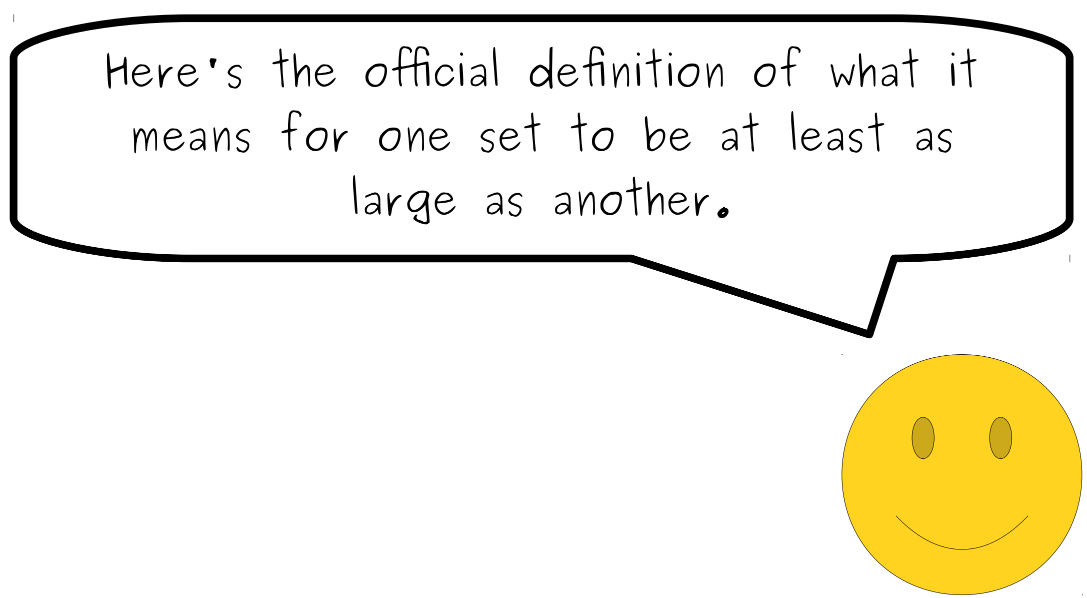
$\mathbb{Z}$ ...   -3   -2   -1   0   1   2   3   ...

$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

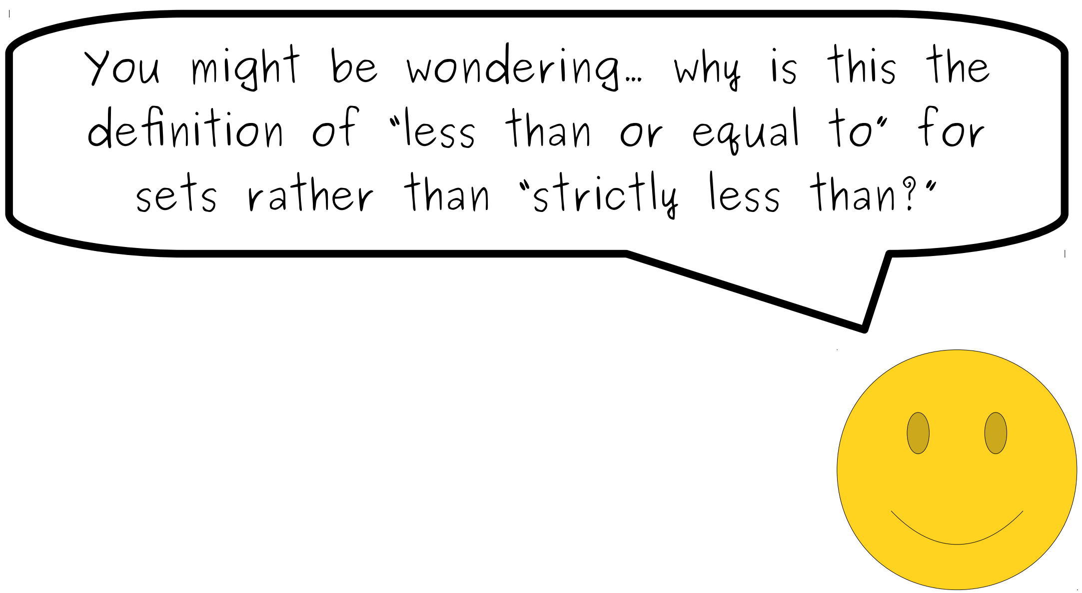


$$\begin{array}{ccccccccccc}
\mathbb{N} & & & & & 0 & 1 & 2 & 3 & \dots \\
 & & & & f : \mathbb{N} \to \mathbb{Z} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Z} & \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \dots
\end{array}$$

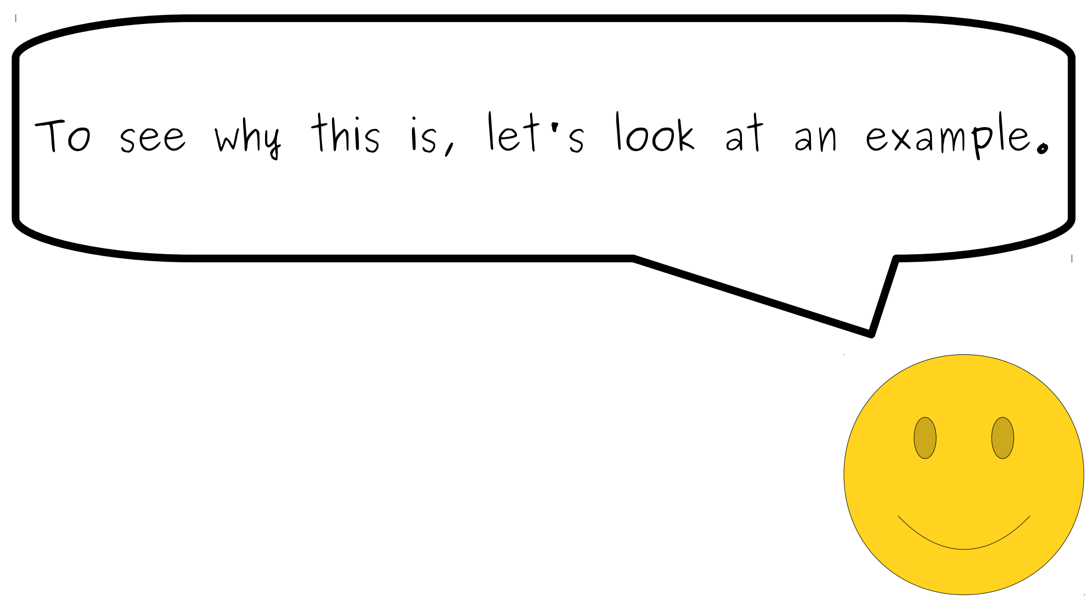
$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$



$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

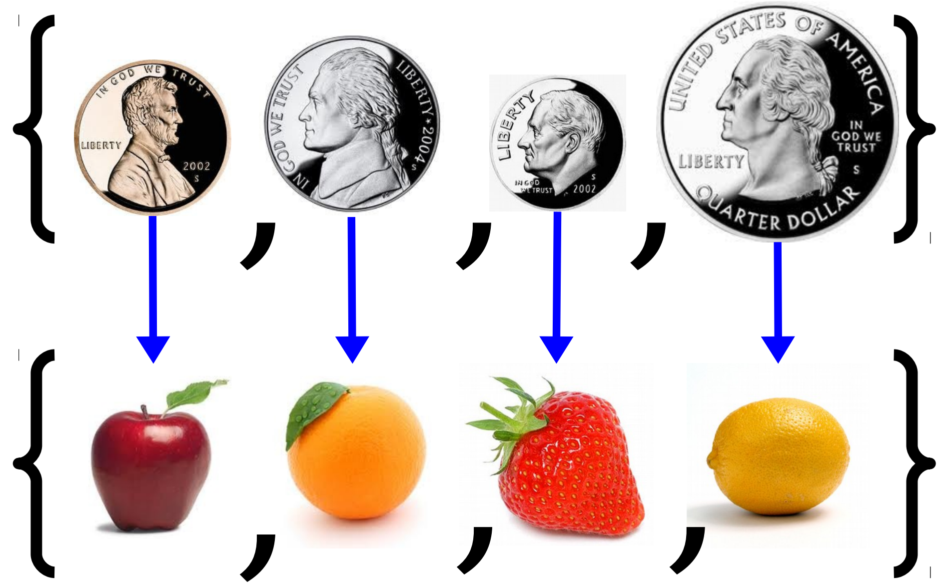


$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$



S

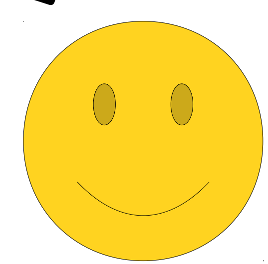
T

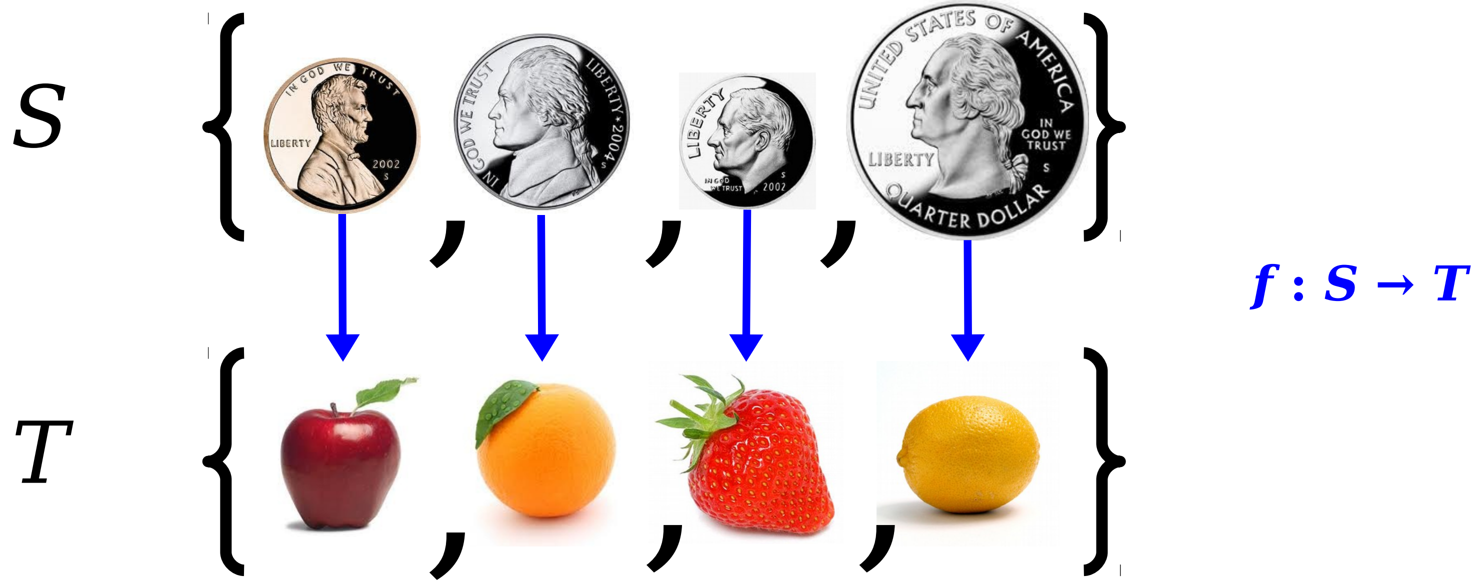


$\boldsymbol{f : S \to T}$

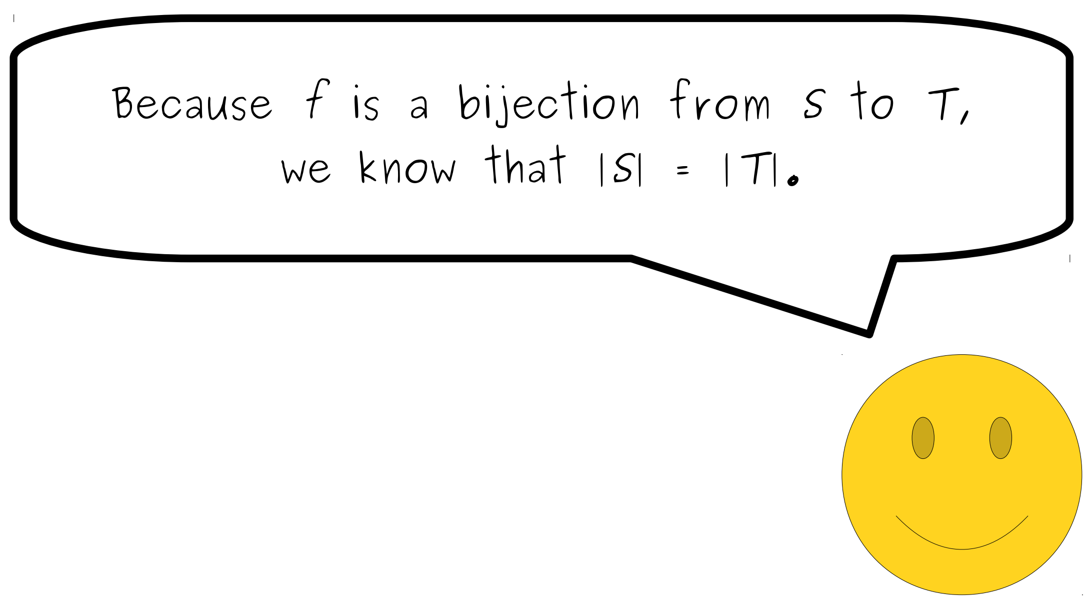
$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

Here's two sets we've seen from before, along with a bijection between them.



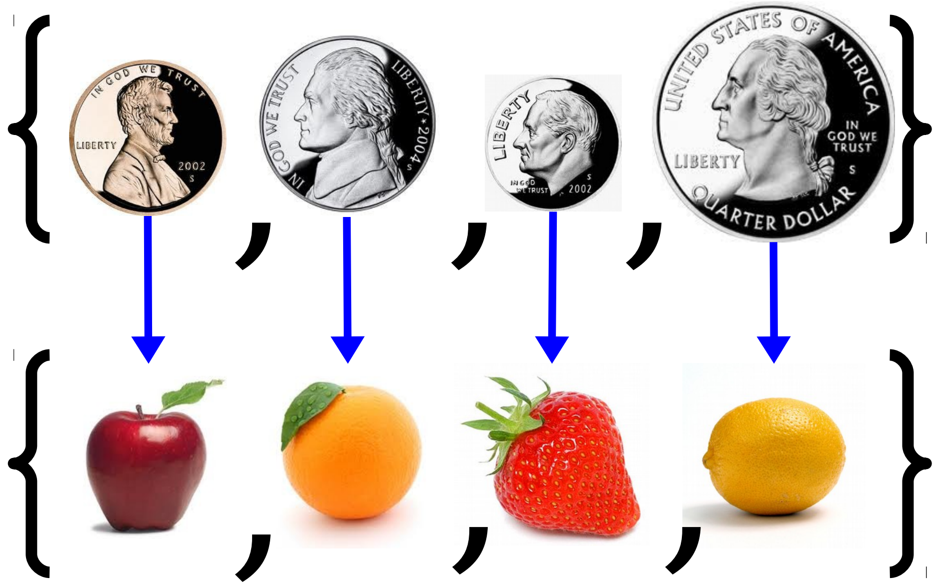


$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$



S

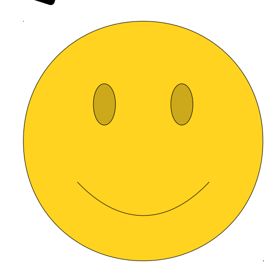
T

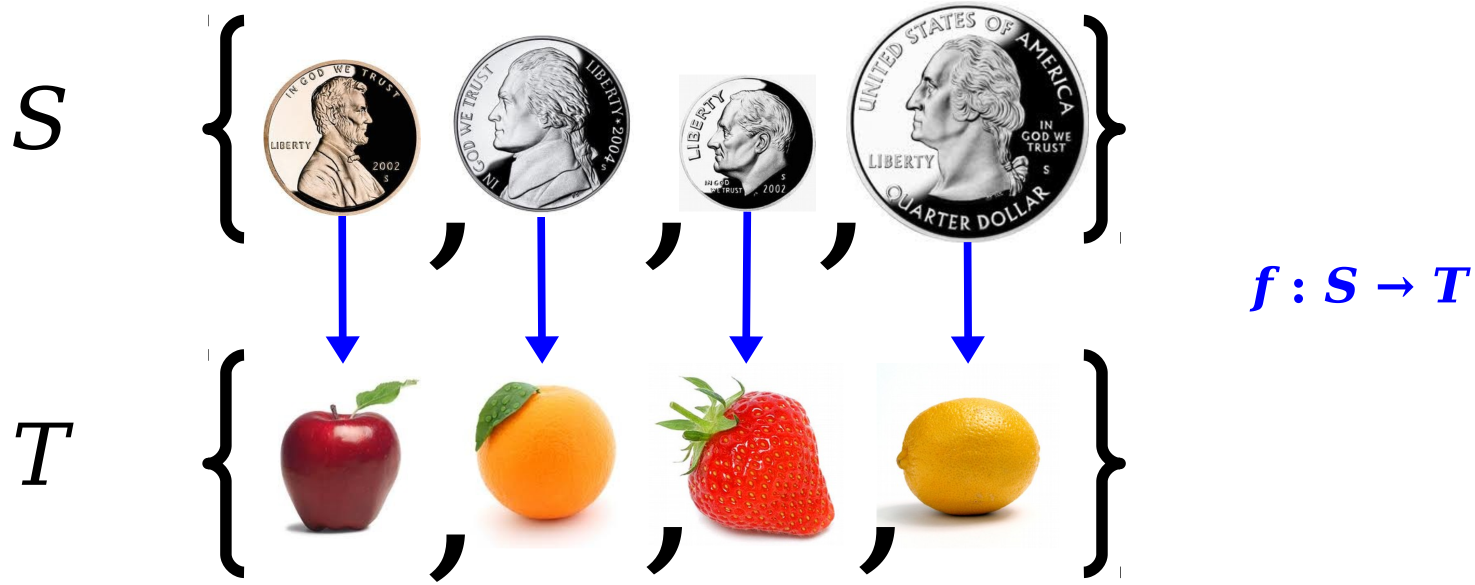


$f : S \to T$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

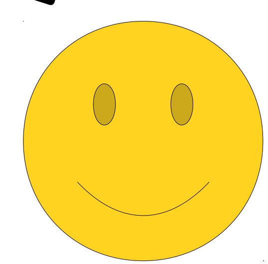
At the same time, this function f is also an injection (remember, all bijections are also injective!)





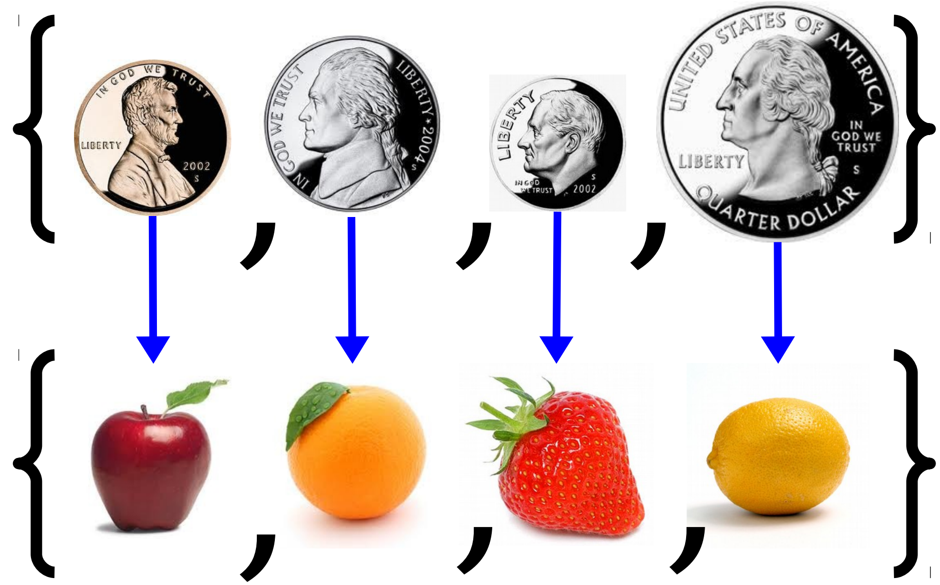
$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

So, according to our definition, this means that $|S| \leq |T|$. That's okay, though – it matches our intuition.



S

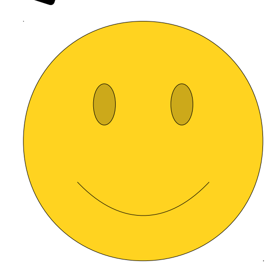
T



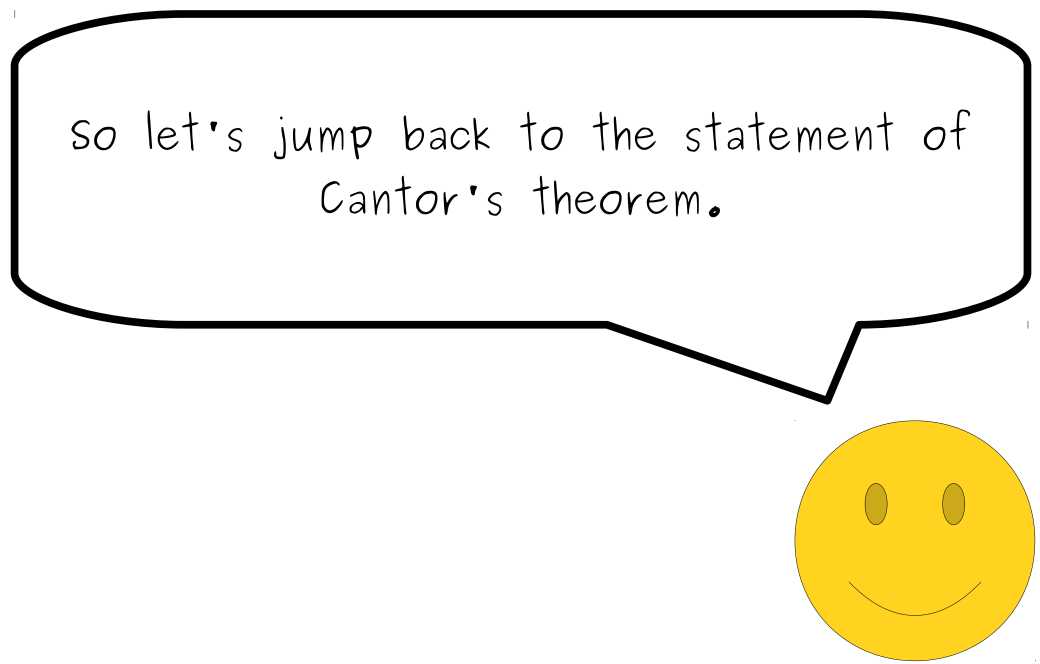
$f : S \to T$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

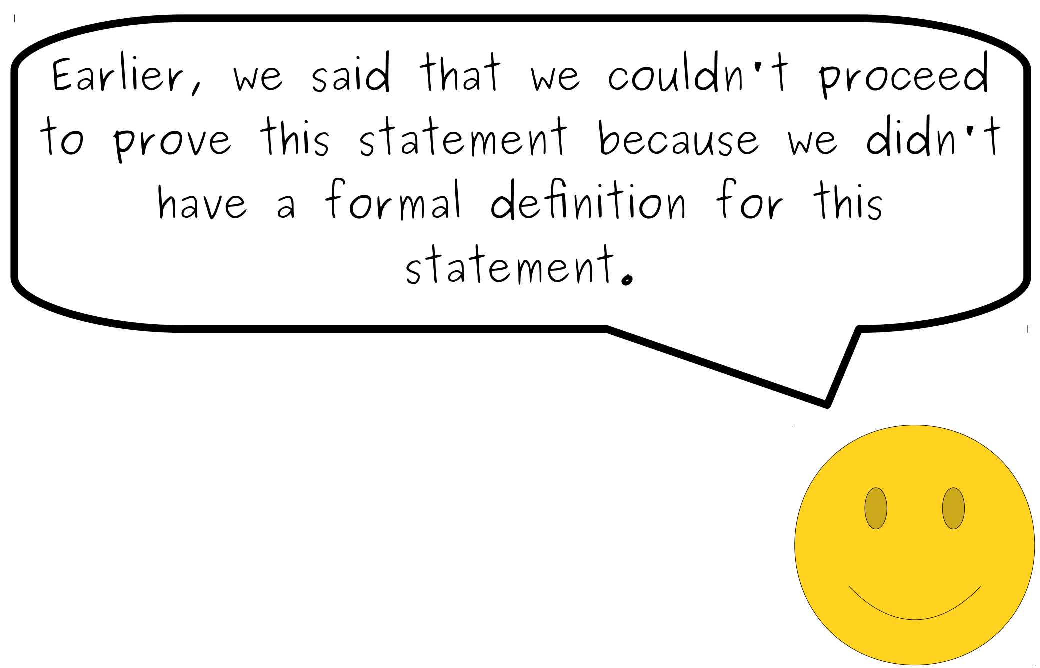
That's why we use this as a definition of $\leq$ over cardinalities – it's to be consistent with our definition of = over cardinalities.



If S is a set, then $|S| < |\wp(S)|$



If S is a set, then $|S| < |\wp(S)|$



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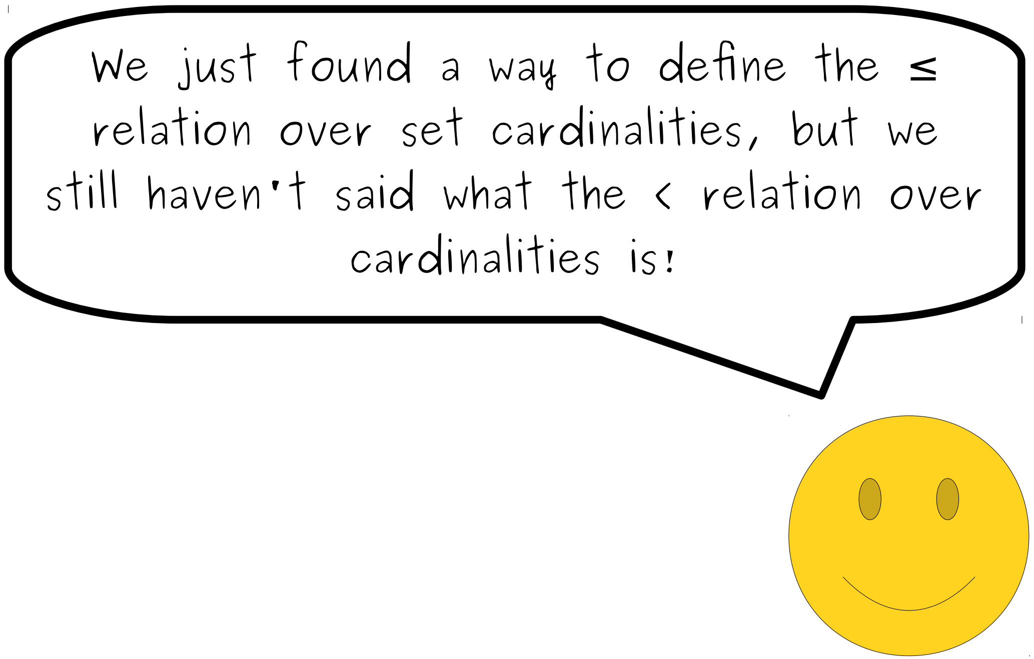
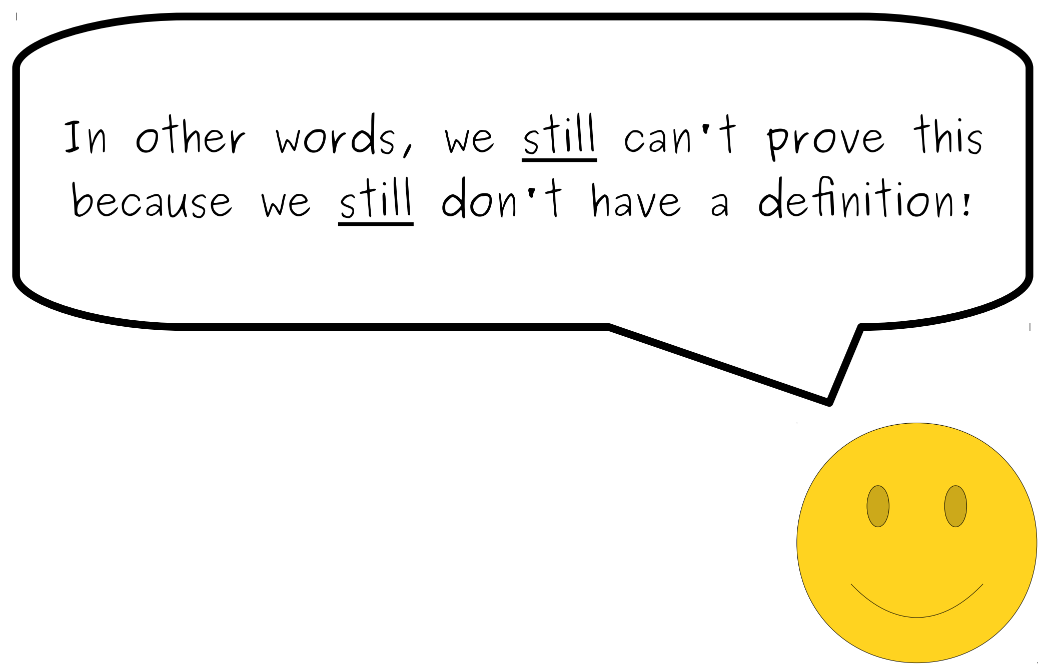
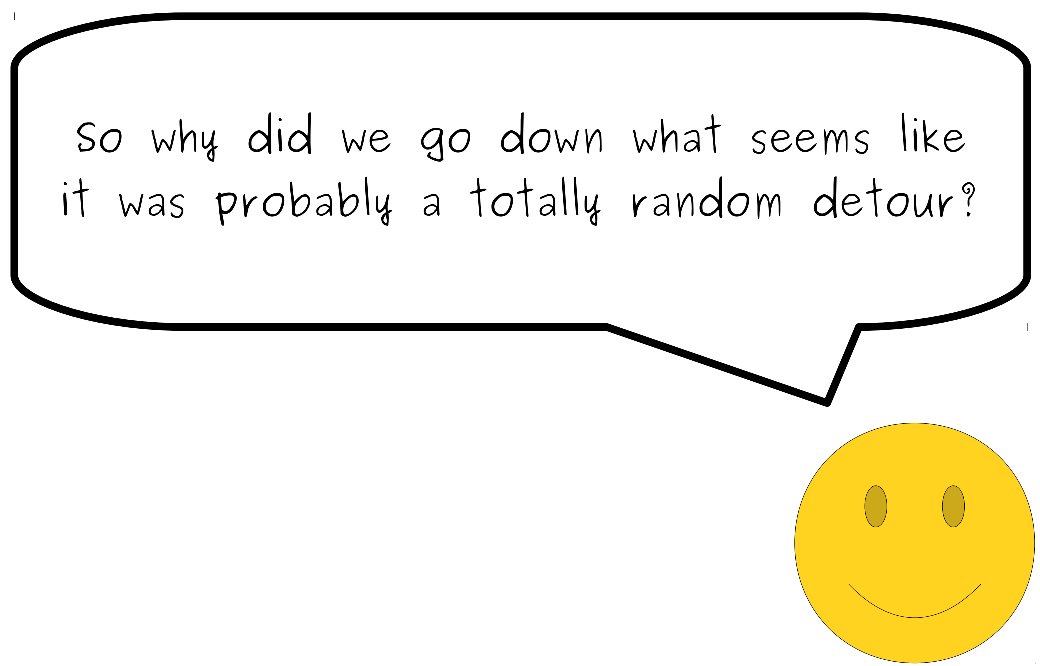


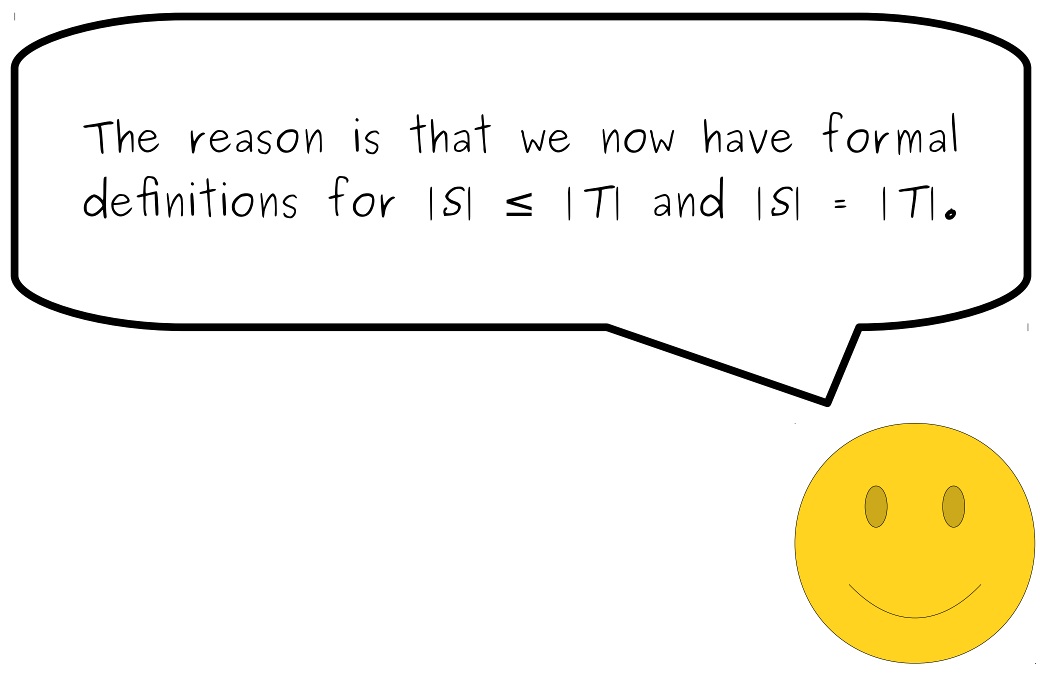
If S is a set, then $|S| < |\wp(S)|$



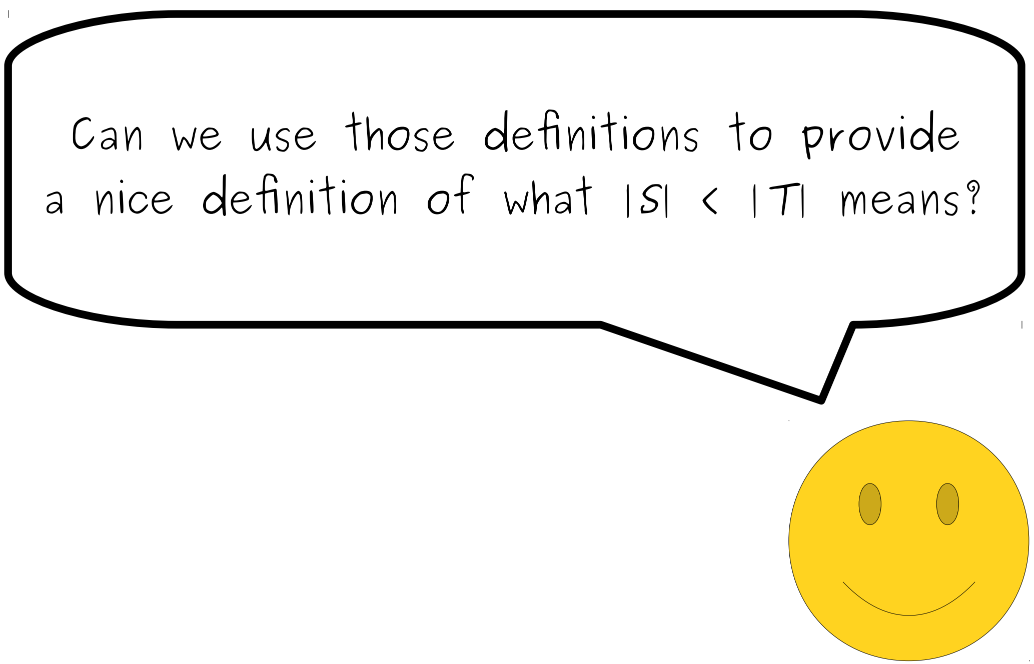
If S is a set, then $|S| < |\wp(S)|$

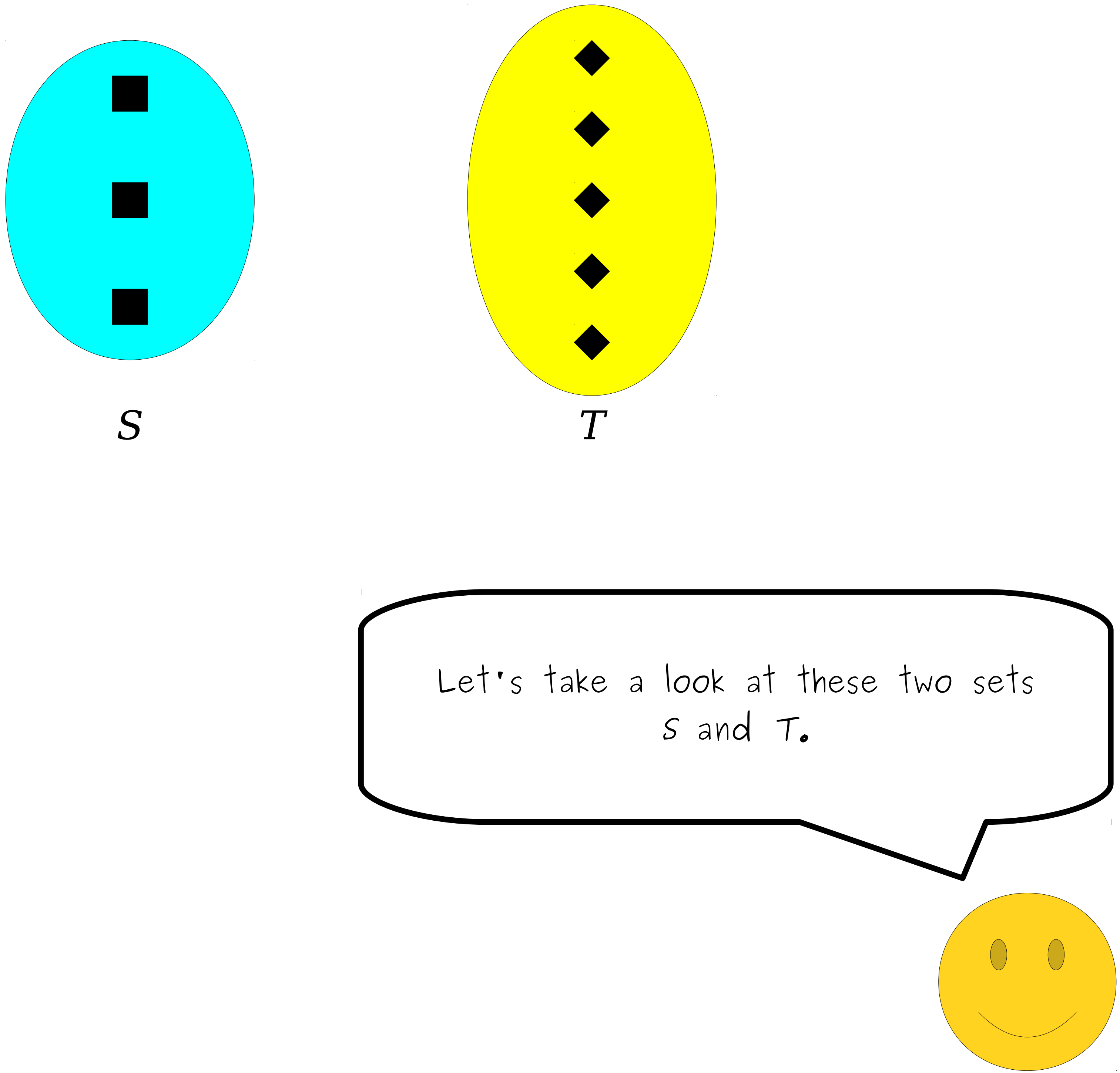


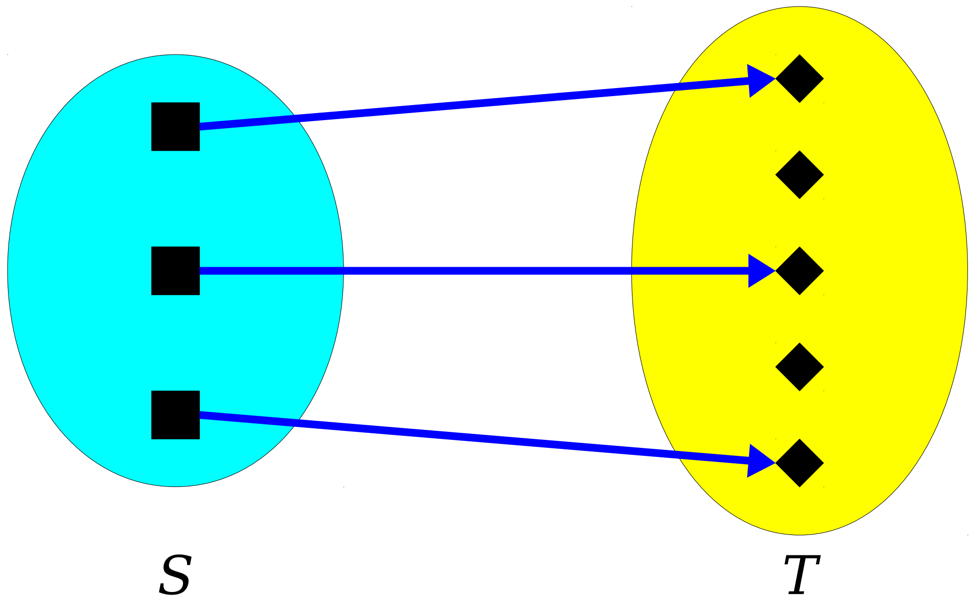
If S is a set, then $|S| < |\wp(S)|$

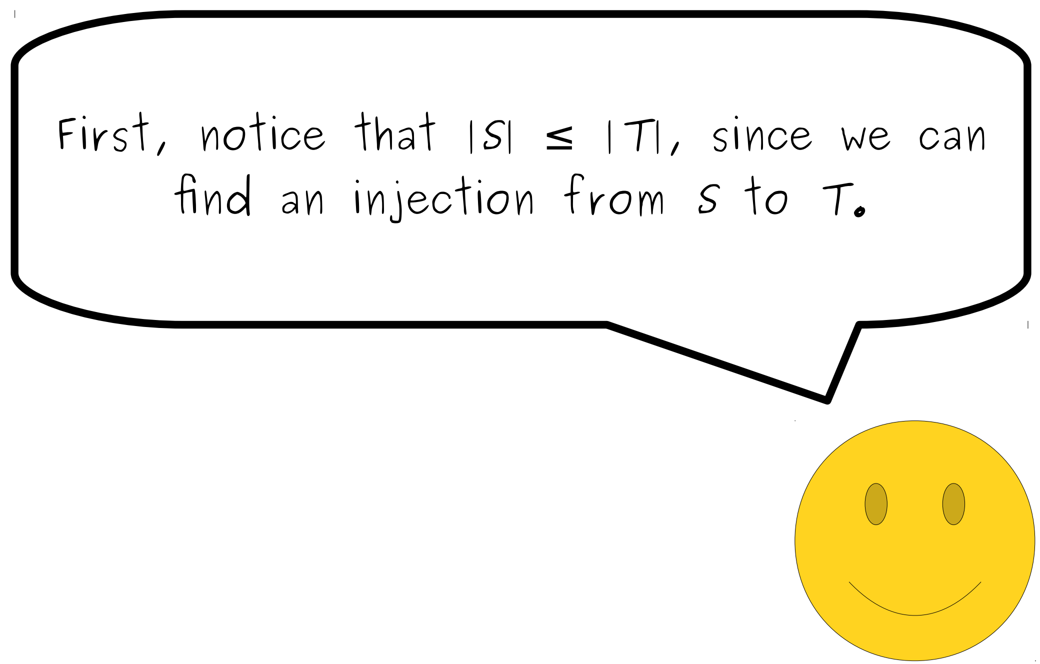


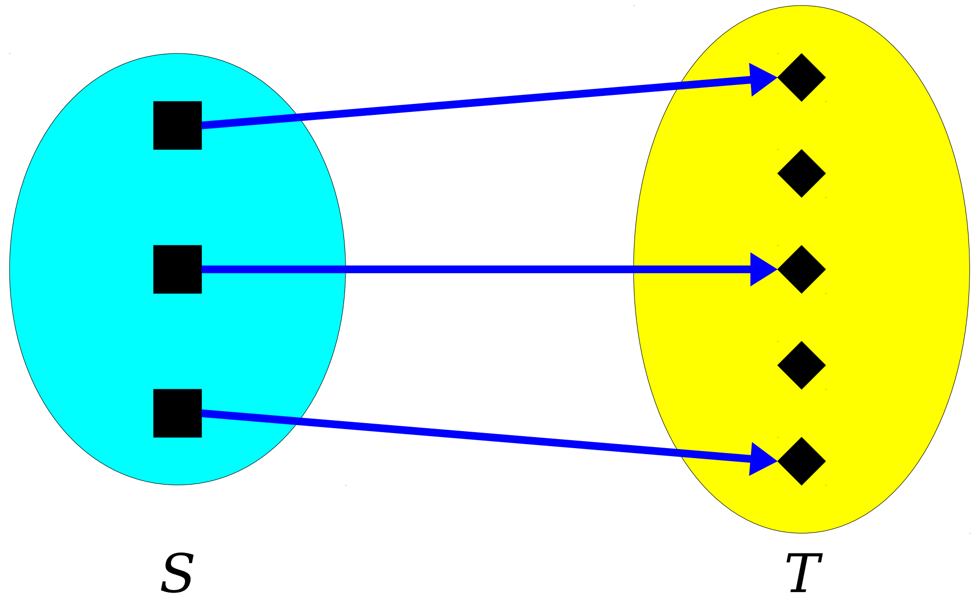
If S is a set, then $|S| < |\wp(S)|$

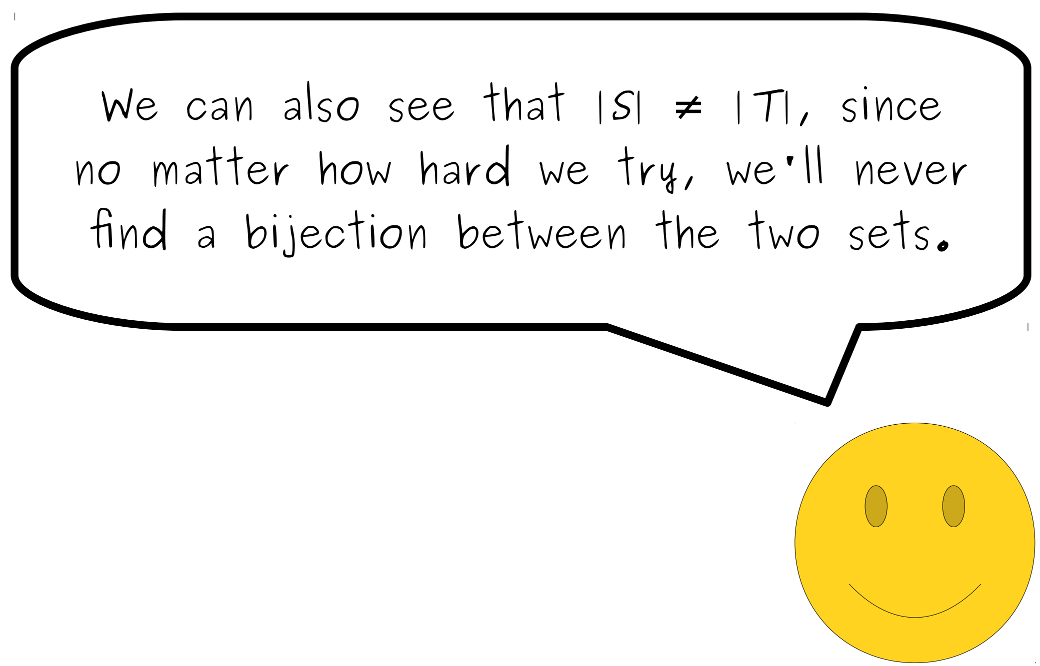


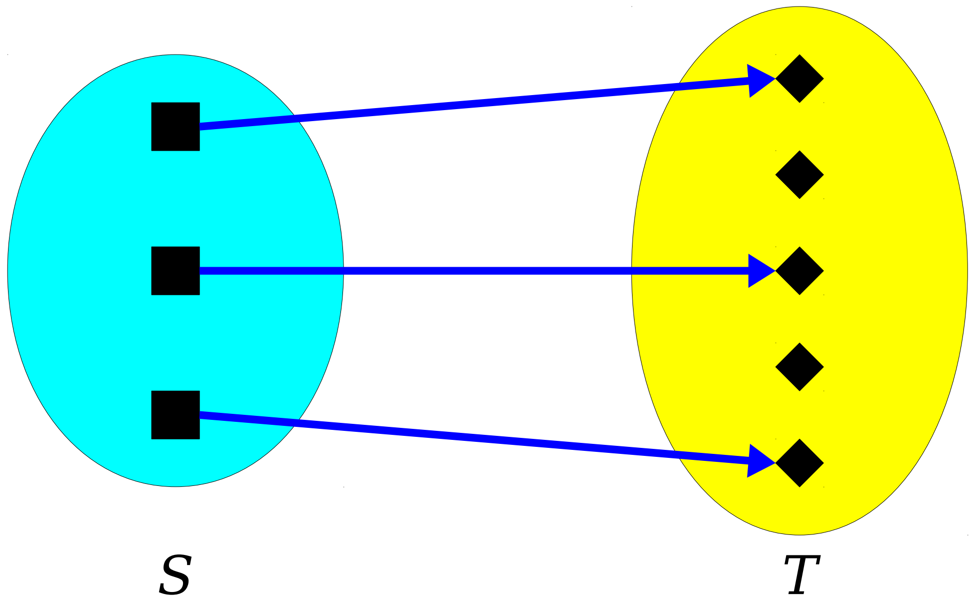

S
T
Let's take a look at these two sets S and T.

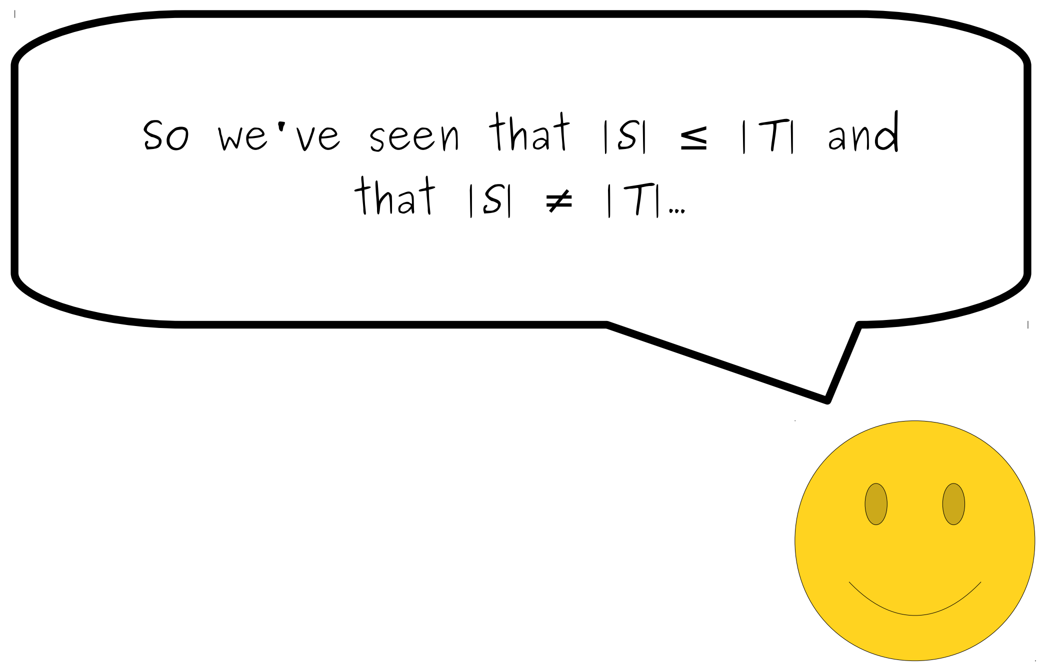

S
T

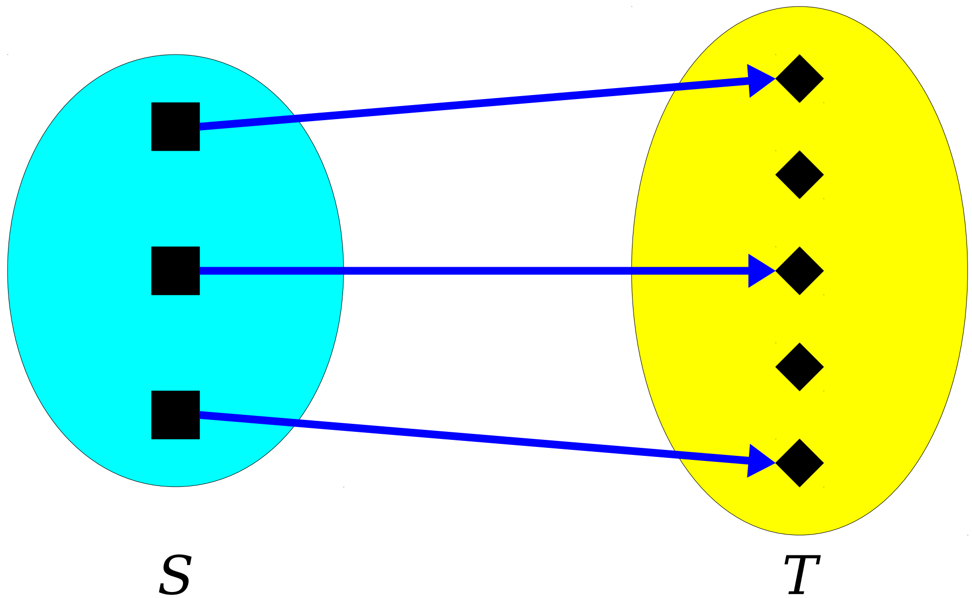

First, notice that $|S| \leq |T|$, since we can find an injection from S to T.

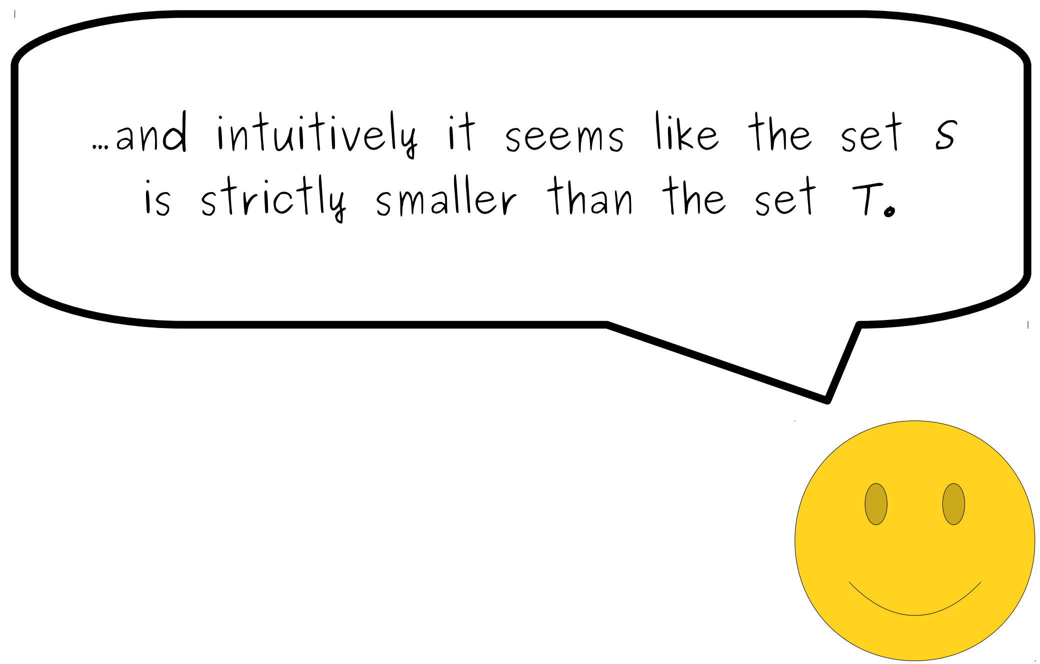

S
T

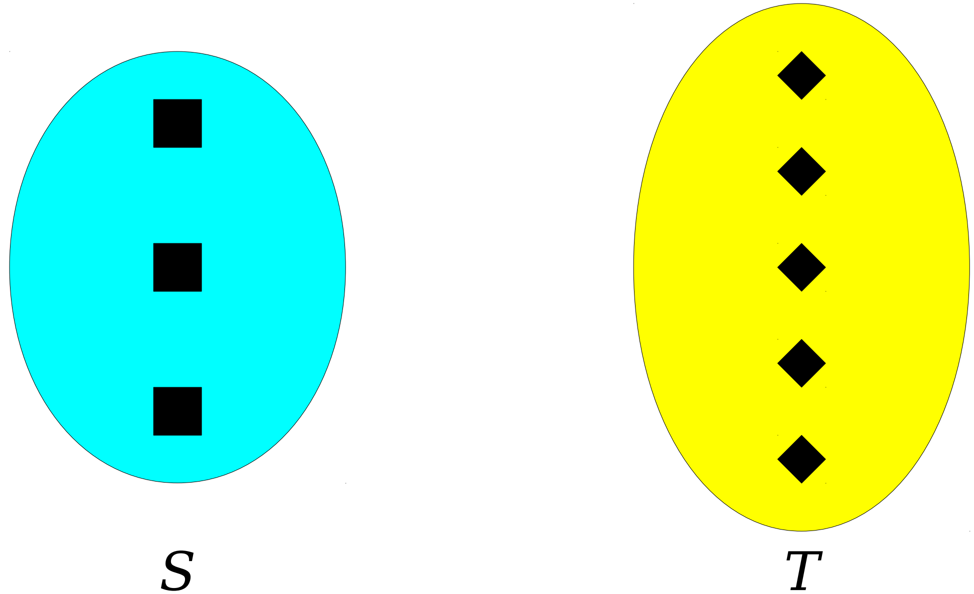

We can also see that |S| ≠ |T|, since no matter how hard we try, we'll never find a bijection between the two sets.

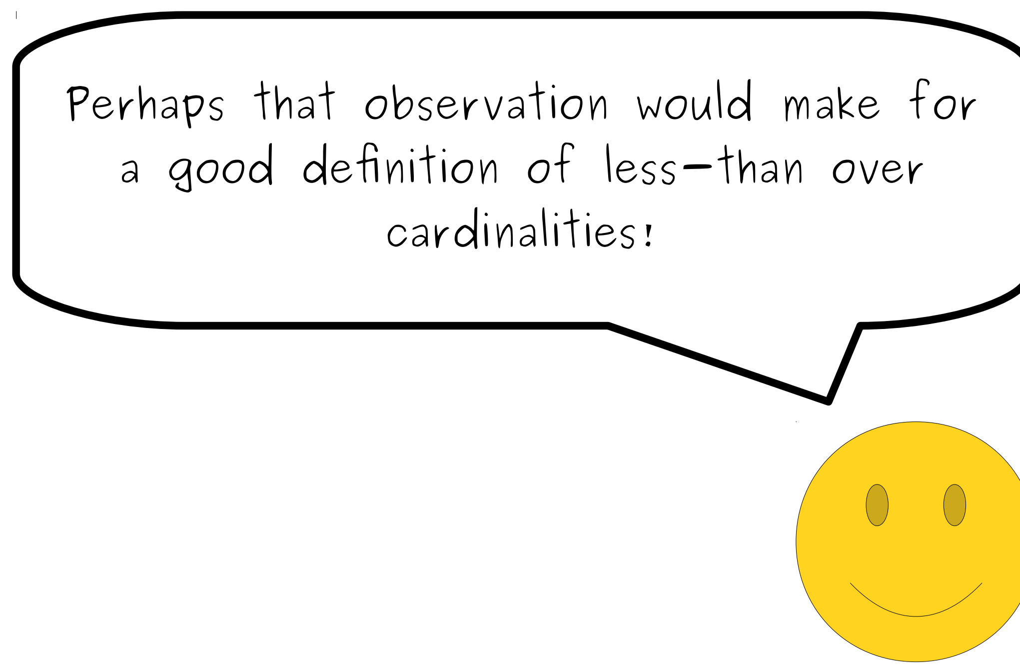

S
T


So we've seen that |S| ≤ |T| and that |S| ≠ |T|...

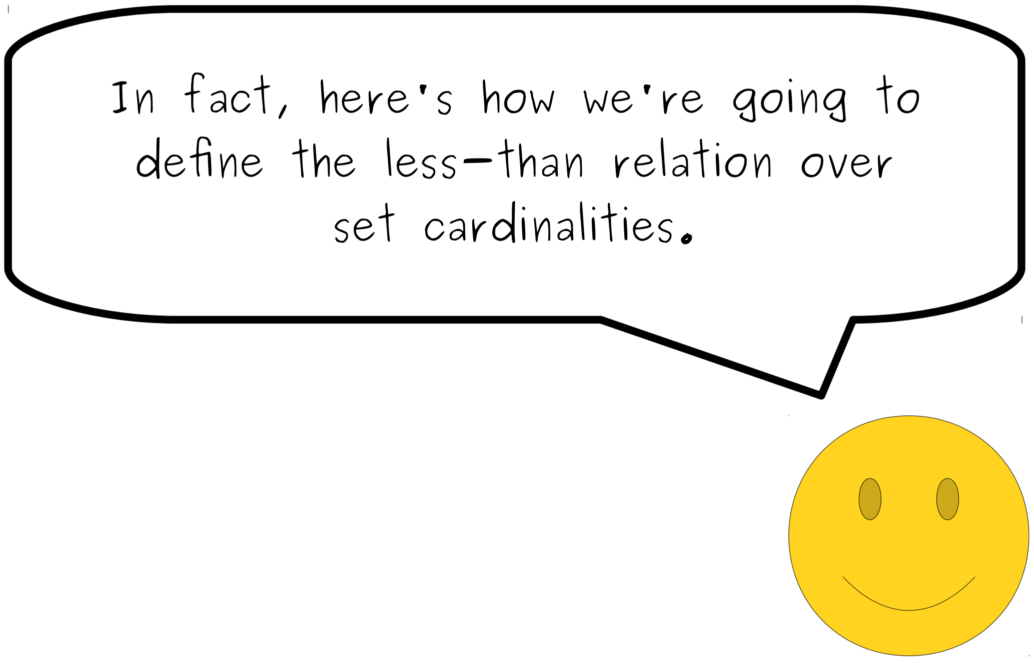

S
T


...and intuitively it seems like the set S is strictly smaller than the set T.

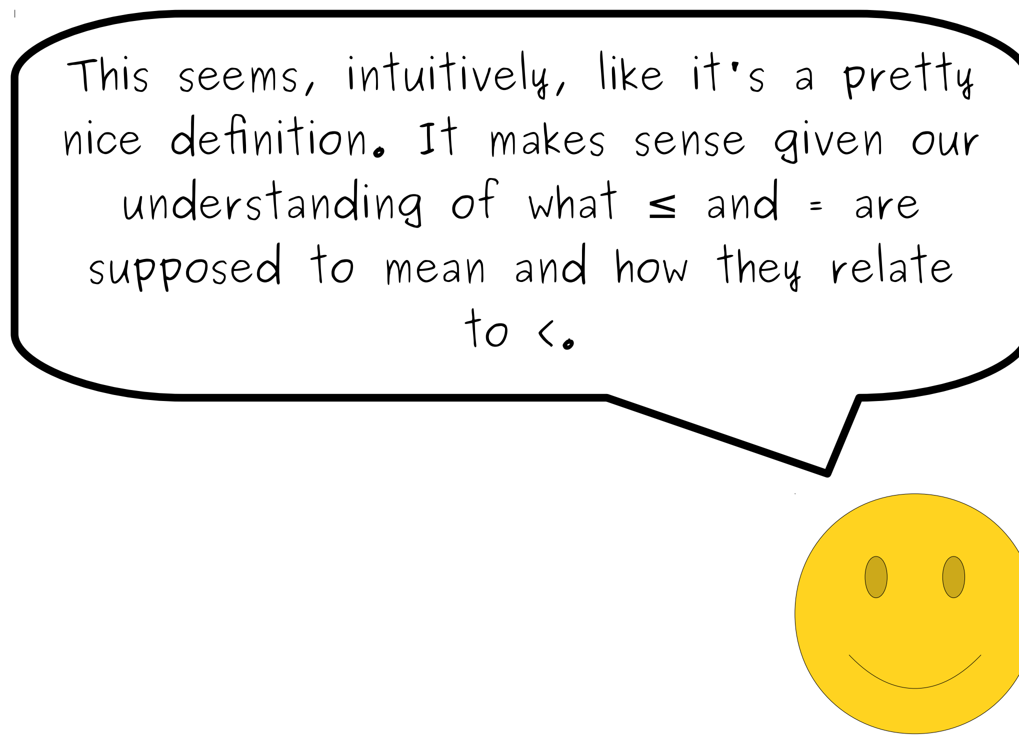

S
T


Perhaps that observation would make for a good definition of less-than over cardinalities!

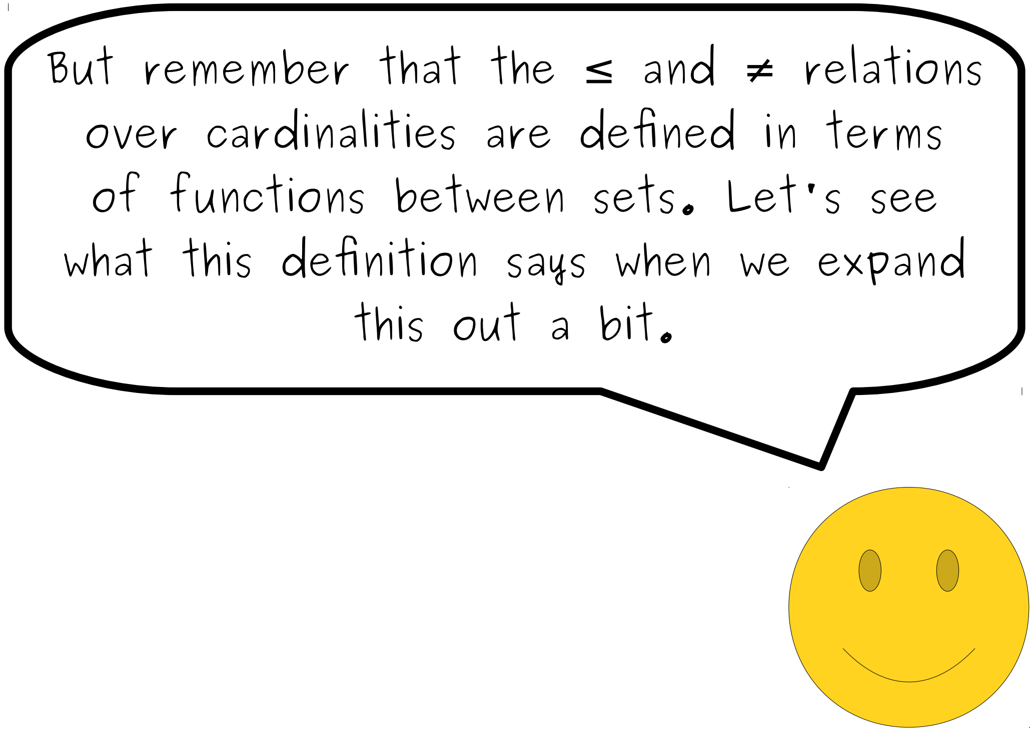
$$|S| < |T| \quad \textbf{if} \quad |S| \leq |T| \textbf{ and } |S| \neq |T|$$



$|S| < |T|$ if $|S| \le |T|$ and $|S| \ne |T|$



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

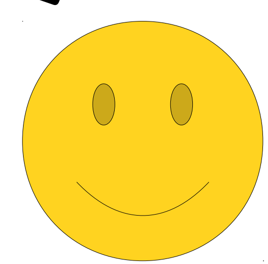


$$|S| < |T| \quad \textbf{if} \quad \underbrace{|S| \leq |T|}_{\text{There is an injective function } f : S \to T} \textbf{ and } |S| \neq |T|$$

This part of the definition says that there should be an injective function from S to T...

$$|S| < |T| \quad \textbf{if} \quad \underbrace{|S| \le |T|}_{\text{There is an injective function } f : S \to T} \textbf{ and } \underbrace{|S| \ne |T|}_{\text{No function } f : S \to T \text{ is a bijection.}}$$

...and this part says that there aren't any bijective functions from S to T.



$$|S| < |T| \quad \text{if} \quad \underbrace{|S| \leq |T|}_{\text{There is an injective function } f : S \to T} \text{ and } \underbrace{|S| \neq |T|}_{\text{No function } f : S \to T \text{ is a bijection.}}$$

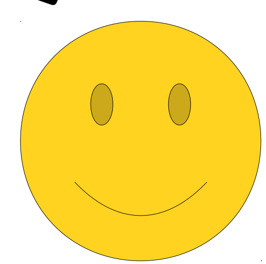
So, if we want to prove that one set is strictly smaller than one another, we should proceed in two parts.

$$|S| < |T| \quad \textbf{if} \quad |S| \le |T| \textbf{ and } |S| \ne |T|$$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

First, we should find an injective function from the smaller set to the larger one.



$$|S| < |T| \quad \textbf{if} \quad \underbrace{|S| \leq |T|}_{\text{There is an injective function } f : S \to T} \textbf{ and } \underbrace{|S| \neq |T|}_{\text{No function } f : S \to T \text{ is a bijection.}}$$

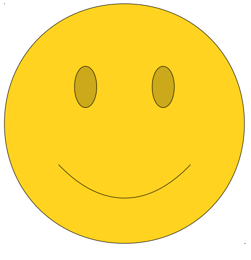
Second, we should prove that there are no bijections from the smaller set to the larger one.

$$|S| < |T| \quad \text{if} \quad |S| \le |T| \text{ and } |S| \ne |T|$$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

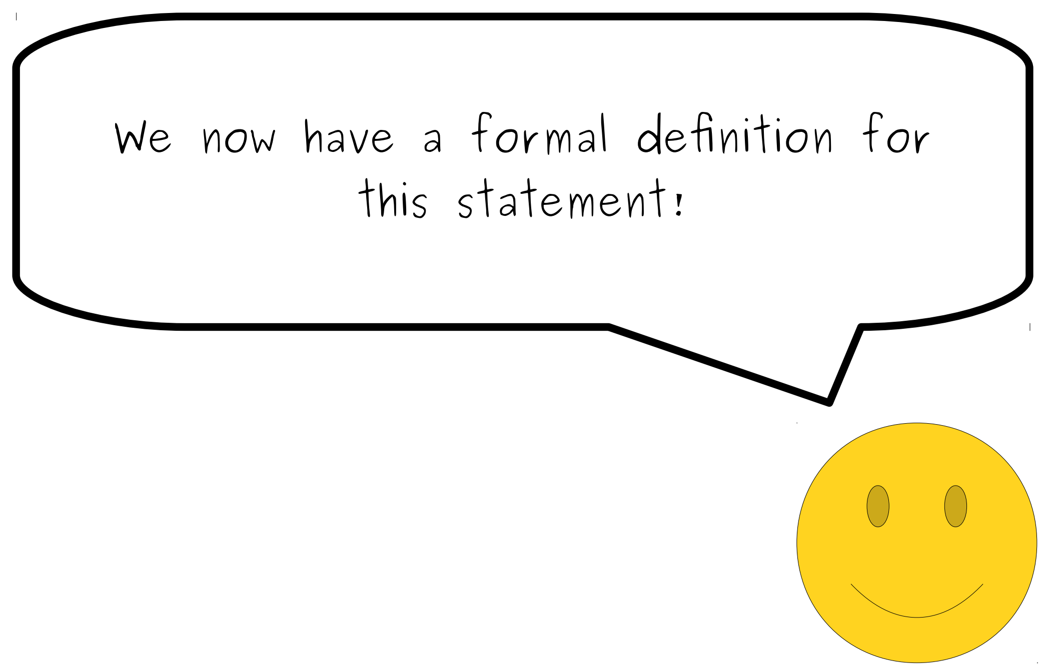
If we can prove both of these statements, then we've shown everything that we've needed to show!



If S is a set, then $|S| < |\wp(S)|$

So let's now take a deeper look at Cantor's theorem.

If S is a set, then $|S| < |\wp(S)|$



There *is* an injection $f : S \to \wp(S)$
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

We now know what we need to prove!
We need to find an injection from
S to $\wp(S)$ and to prove that
there's no bijection from S to $\wp(S)$.

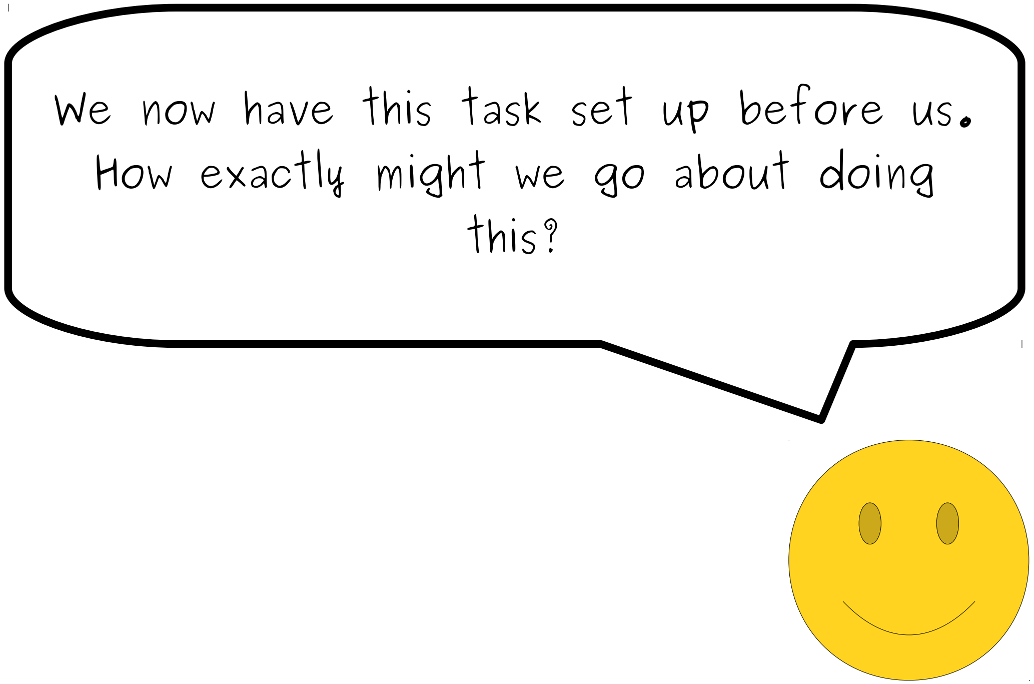
There *is* an injection $f : S \to \wp(S)$

There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

Let's start with this part. Turns out, it's not too bad!

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



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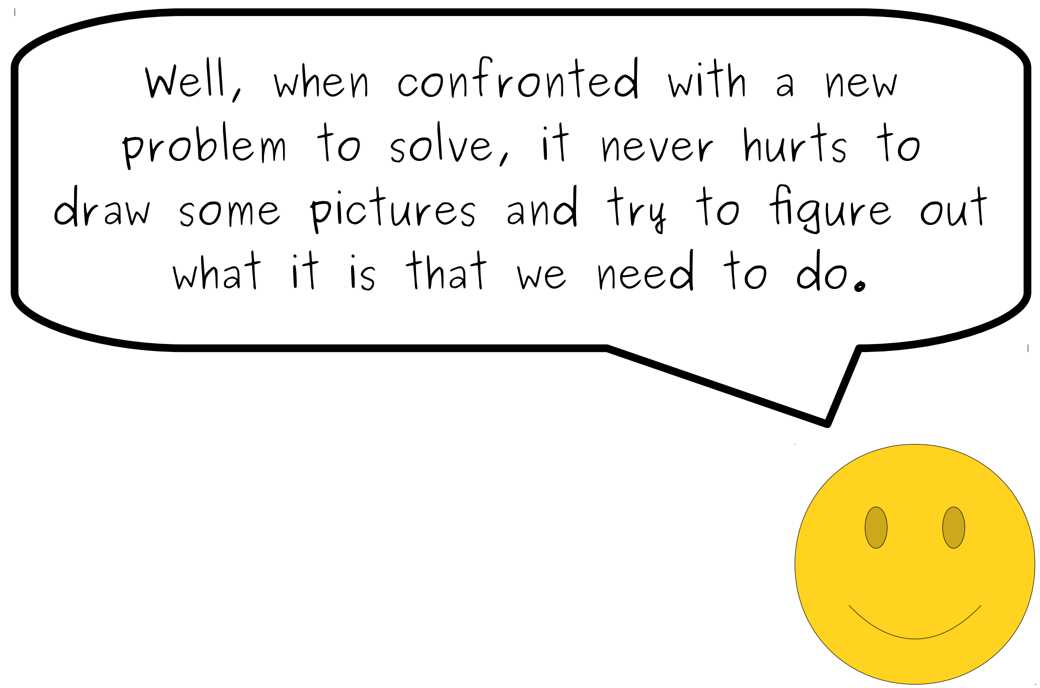

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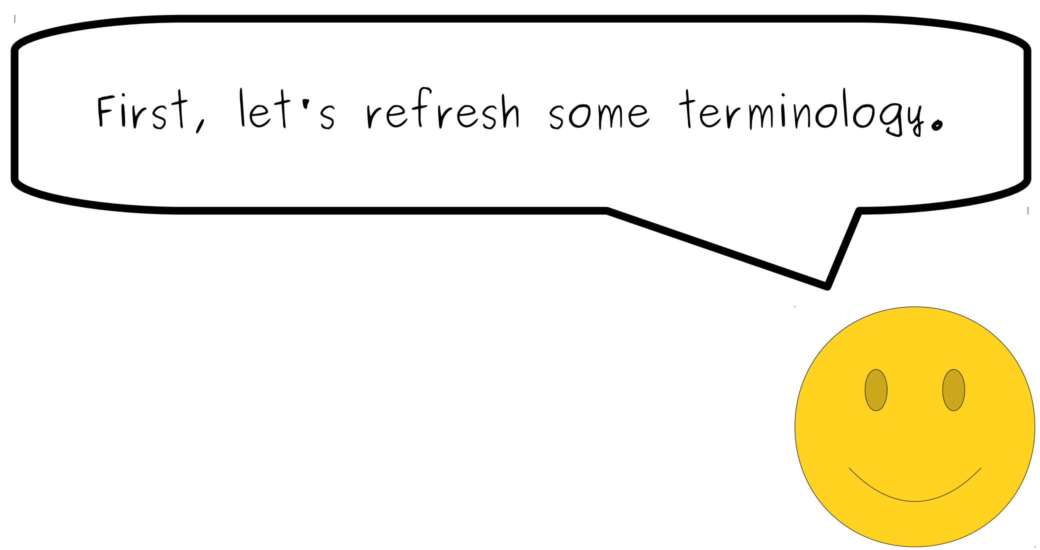
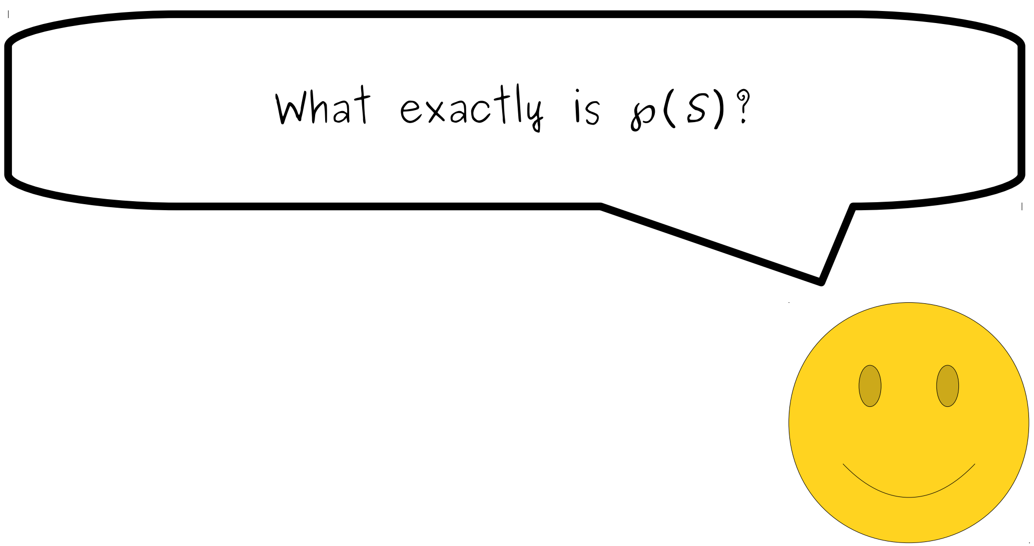
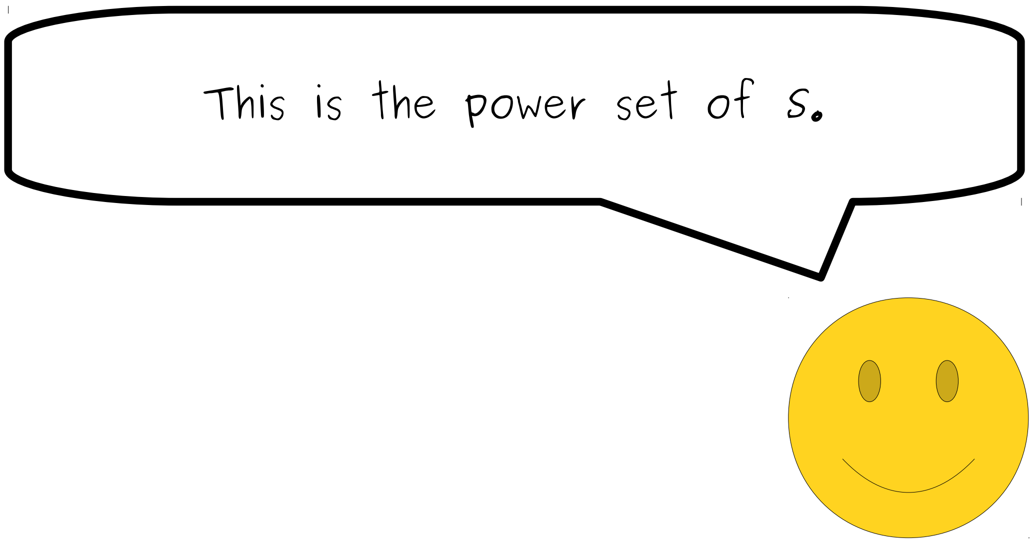


Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



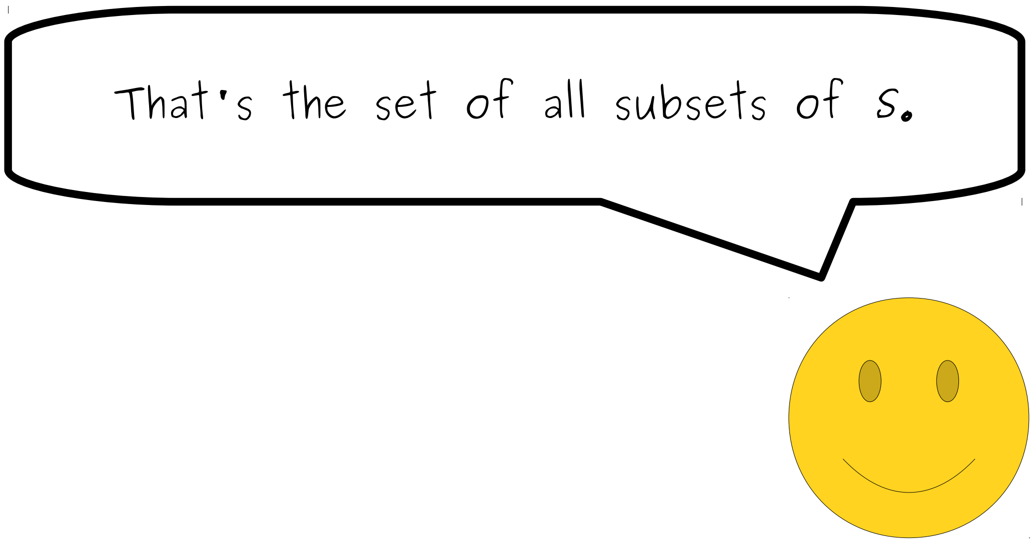
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

The power set of S.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

The power set of S.
The set of all subsets of S.

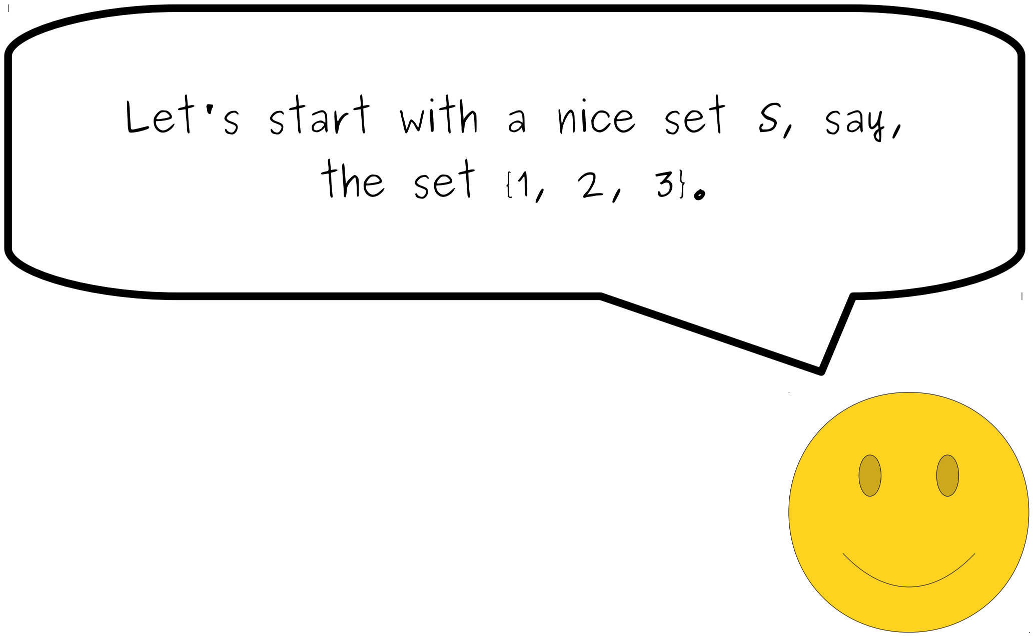


Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

The power set of S.
The set of all subsets of S.

So if our goal is to find an injection from S to $\wp(S)$, we're looking for an injection from the set S to the set of all subsets of S.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



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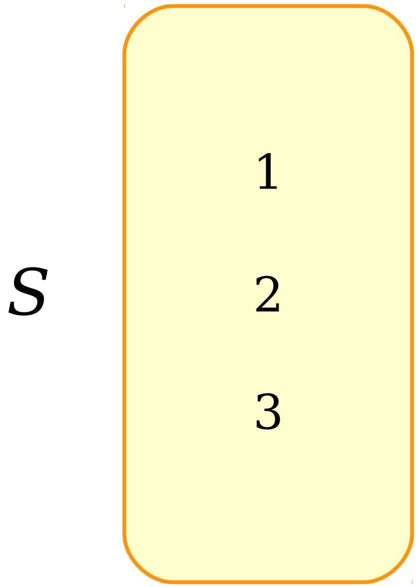
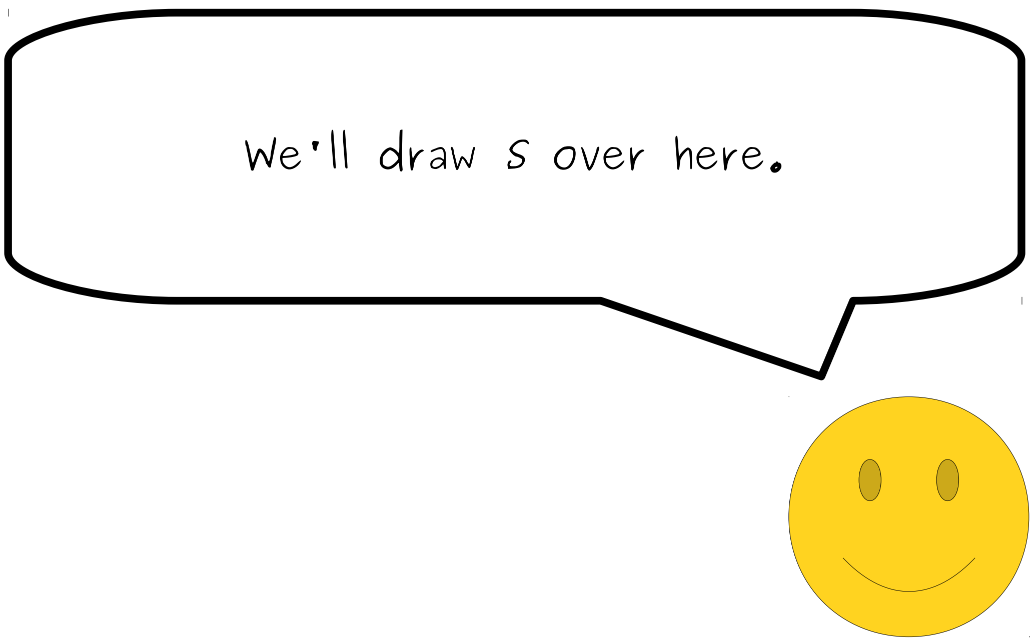




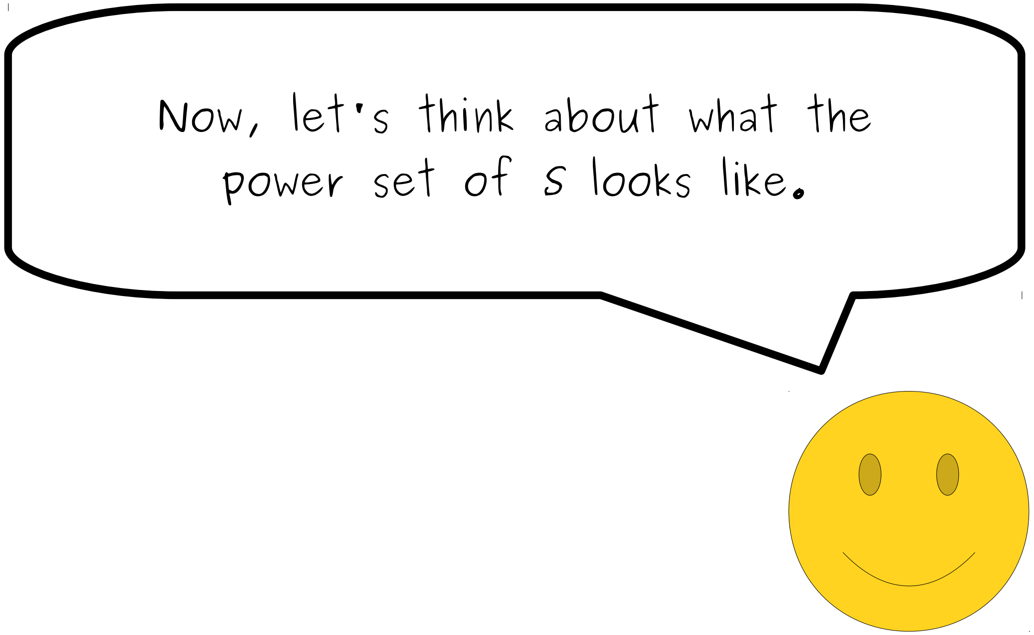
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

S

1

2

3



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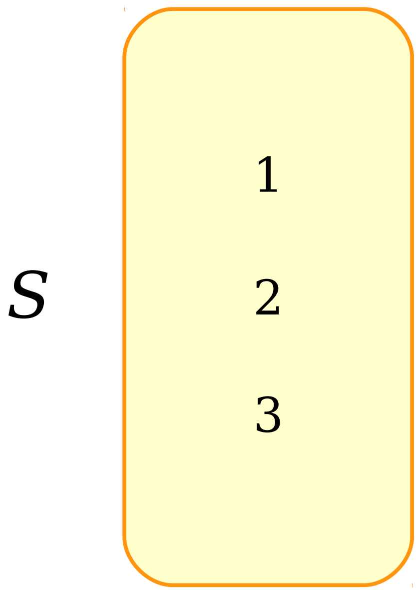

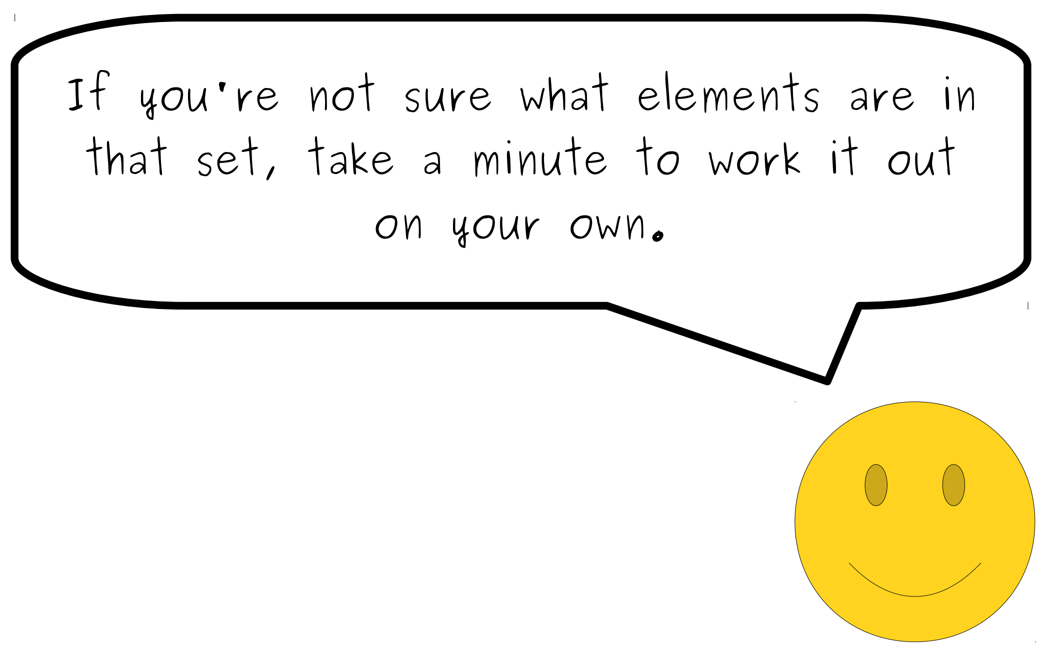

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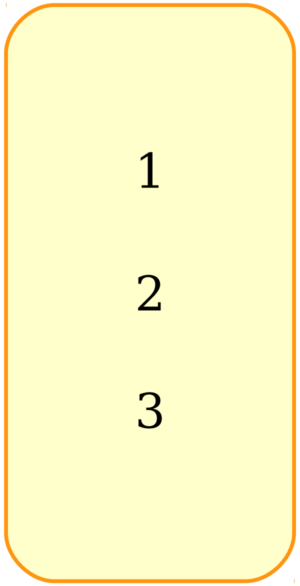
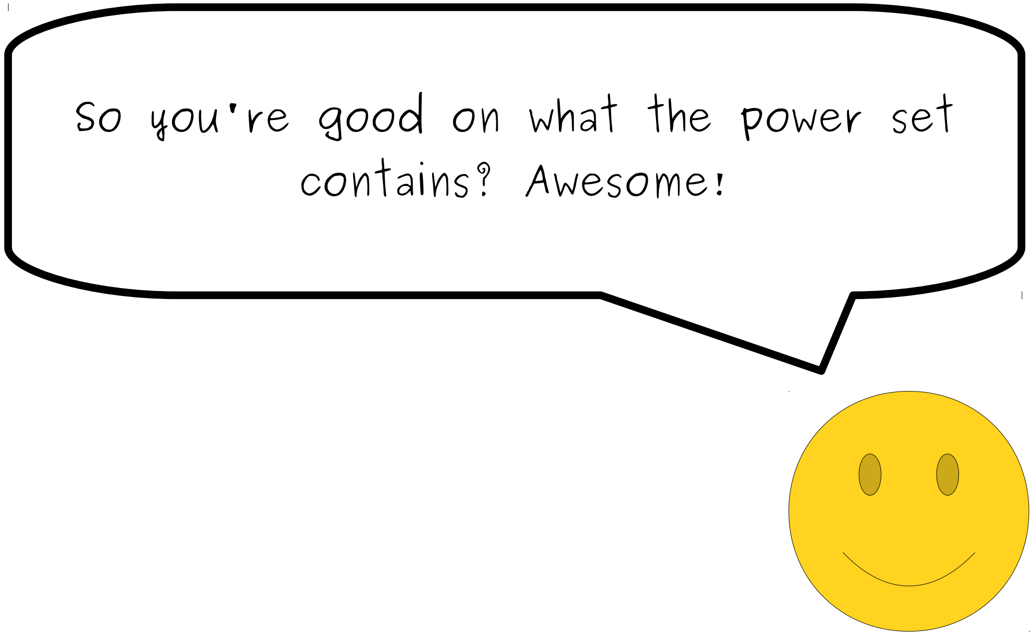
S





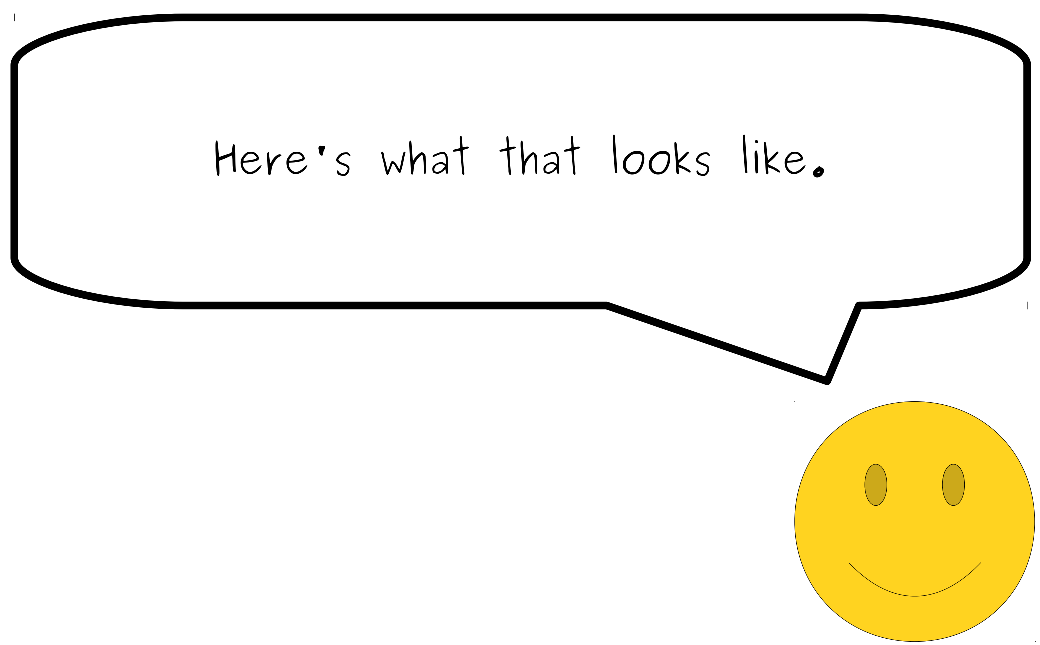
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

S

1
2
3

$\emptyset$
$\{1\}$
$\{2\}$
$\{3\}$
$\{1, 2\}$
$\{1, 3\}$
$\{2, 3\}$
$\{1, 2, 3\}$

$\wp(S)$



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S

1

2

3

$\emptyset$

$\{1\}$

$\{2\}$

$\{3\}$

$\{1, 2\}$

$\{1, 3\}$

$\{2, 3\}$

$\{1, 2, 3\}$

$\wp(S)$

Now, we need to find an injection here from S to $\wp(S)$.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

S

1

2

3

$\wp(S)$

$\emptyset$

$\{1\}$

$\{2\}$

$\{3\}$

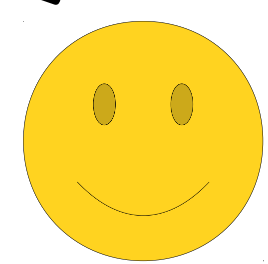
$\{1, 2\}$

$\{1, 3\}$

$\{2, 3\}$

$\{1, 2, 3\}$

Remember our intuition from before – we want to pair off the elements of S with the elements of $\wp(S)$ so that we use up all the elements of S.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

S

1

2

3

$\wp(S)$

$\emptyset$

$\{1\}$

$\{2\}$

$\{3\}$

$\{1, 2\}$

$\{1, 3\}$

$\{2, 3\}$

$\{1, 2, 3\}$

It's okay if some elements of $\wp(S)$ are left over – after all, we're ultimately trying to prove that $|S| < |\wp(S)|$!

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

S

1

2

3

$\wp(S)$

$\varnothing$

$\{1\}$

$\{2\}$

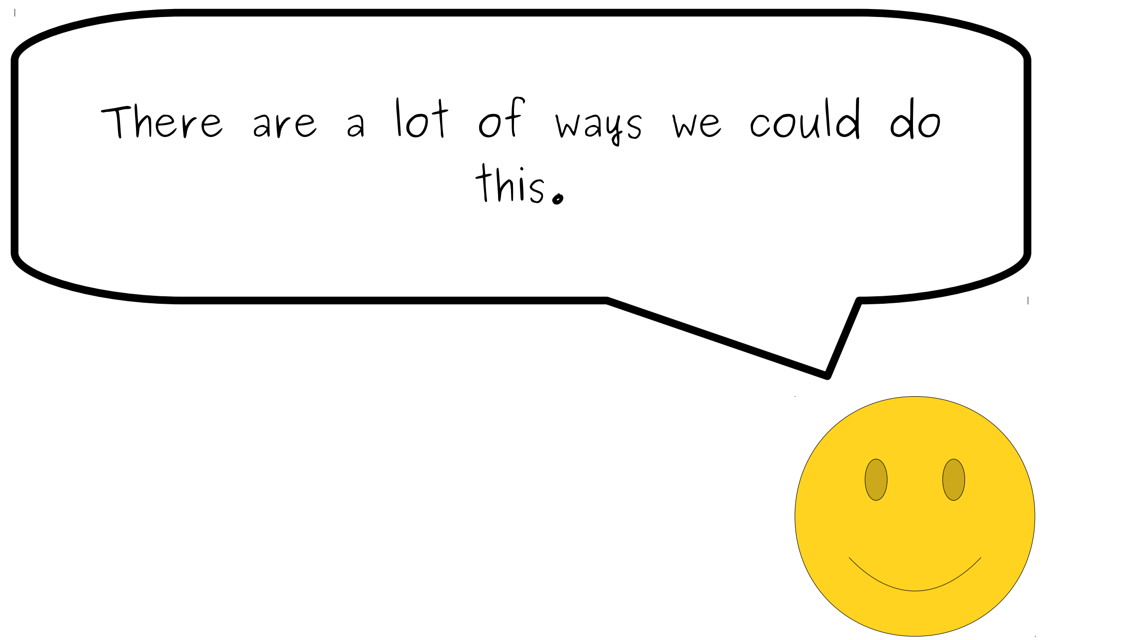
$\{3\}$

$\{1, 2\}$

$\{1, 3\}$

$\{2, 3\}$

$\{1, 2, 3\}$



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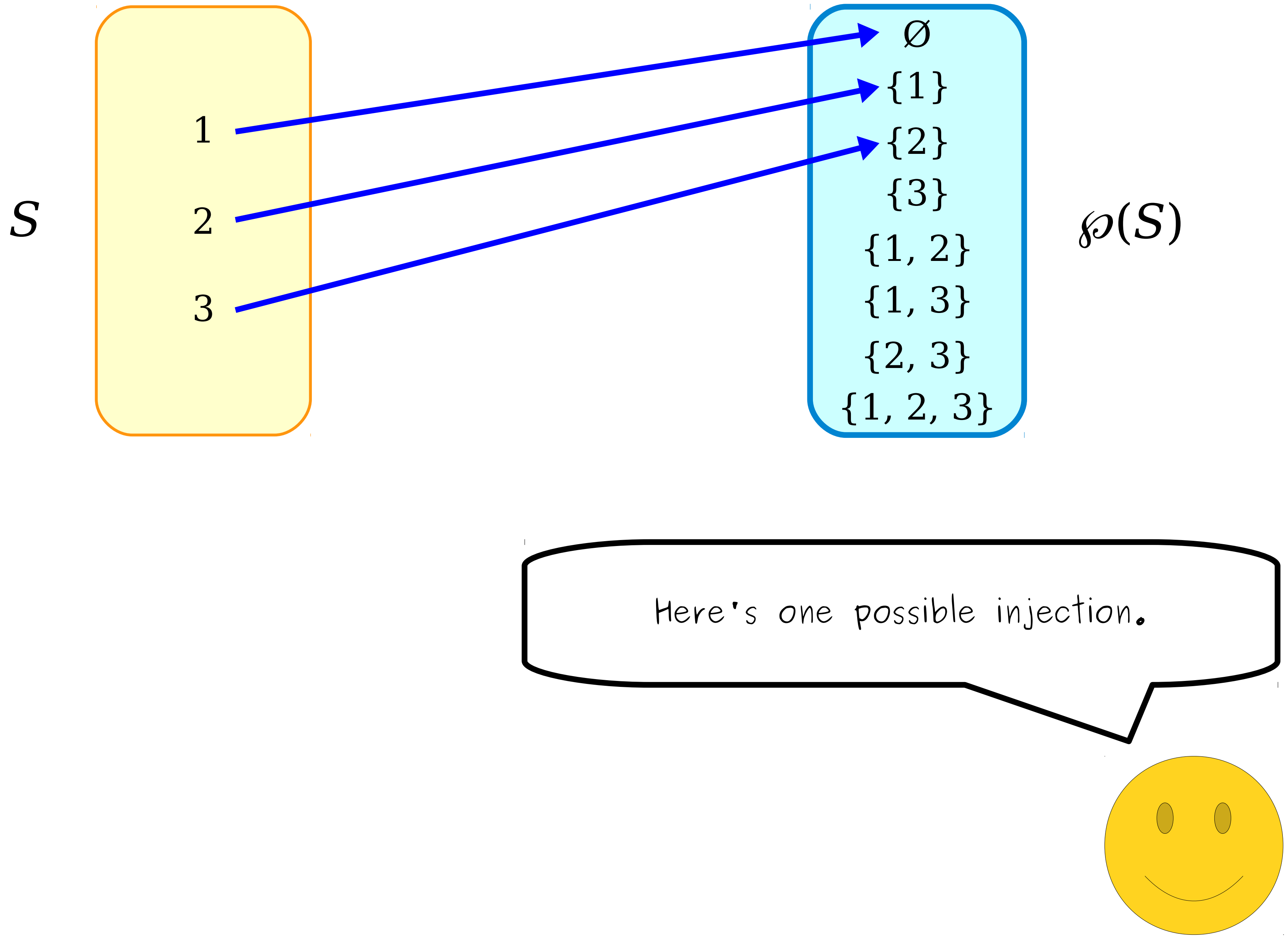

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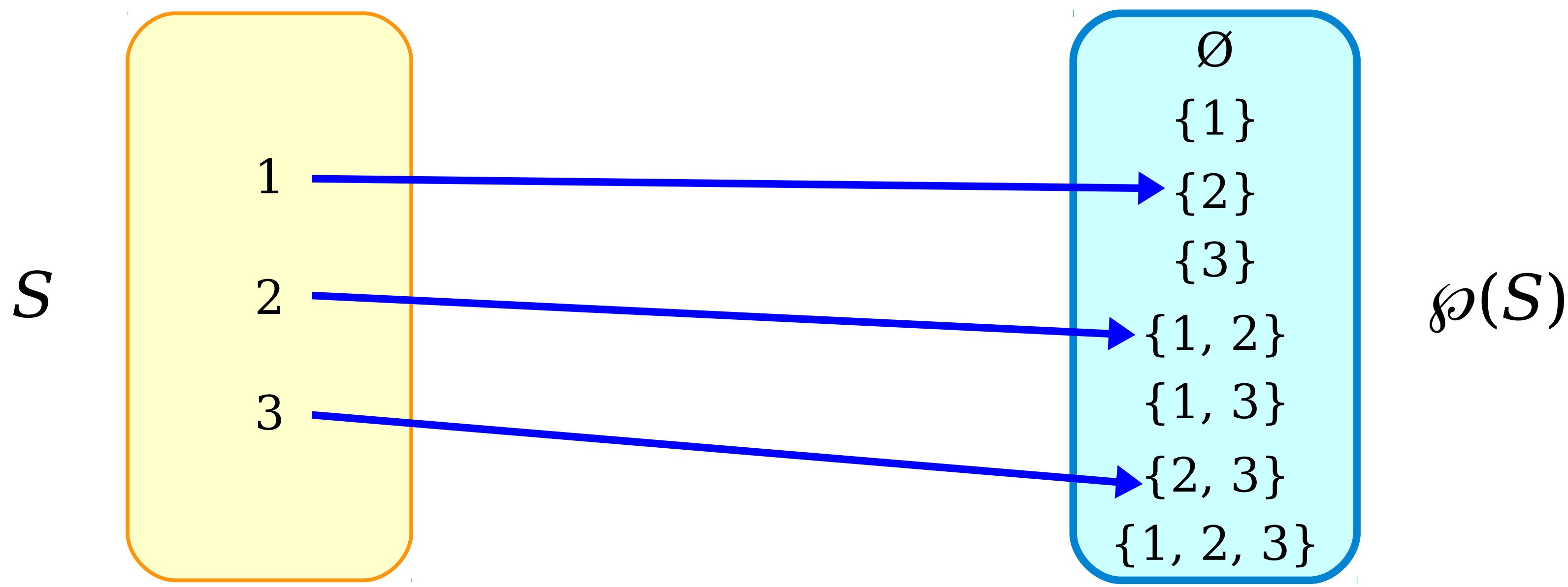

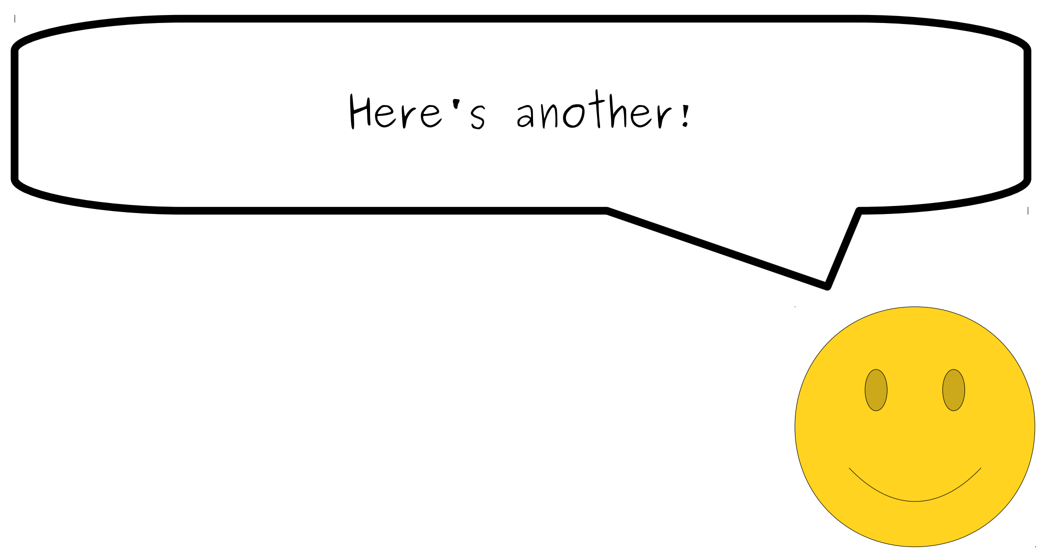

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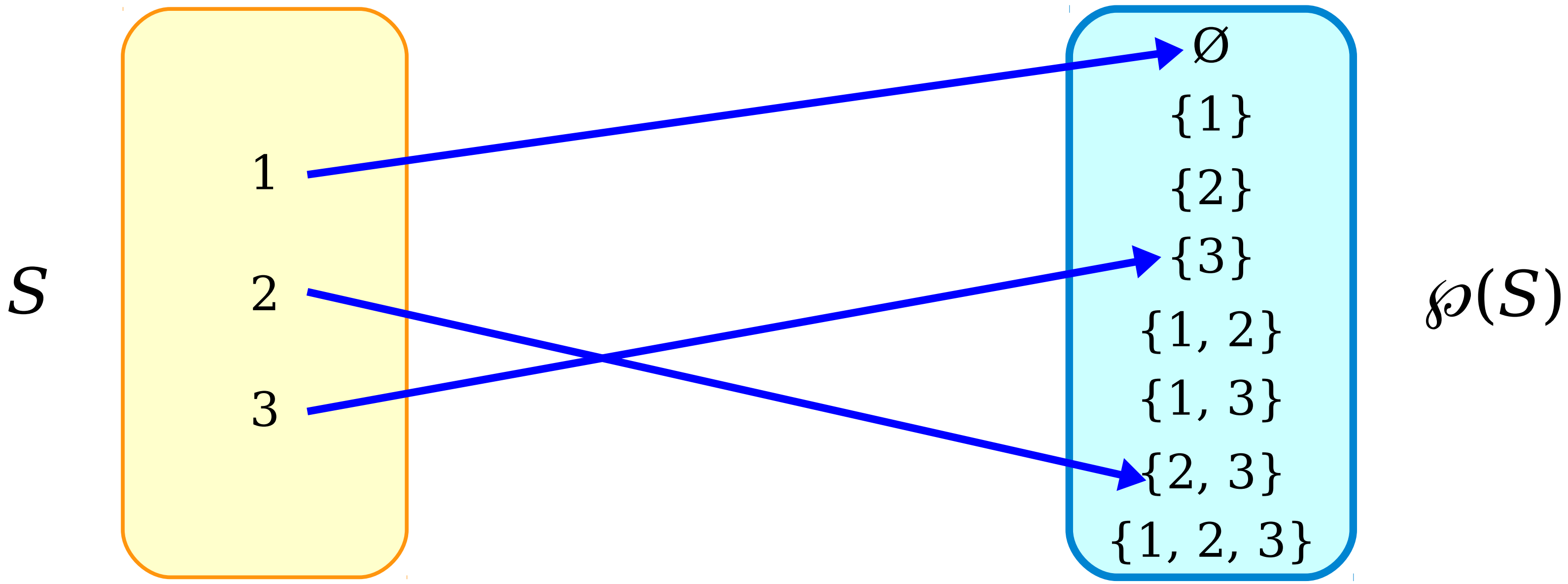

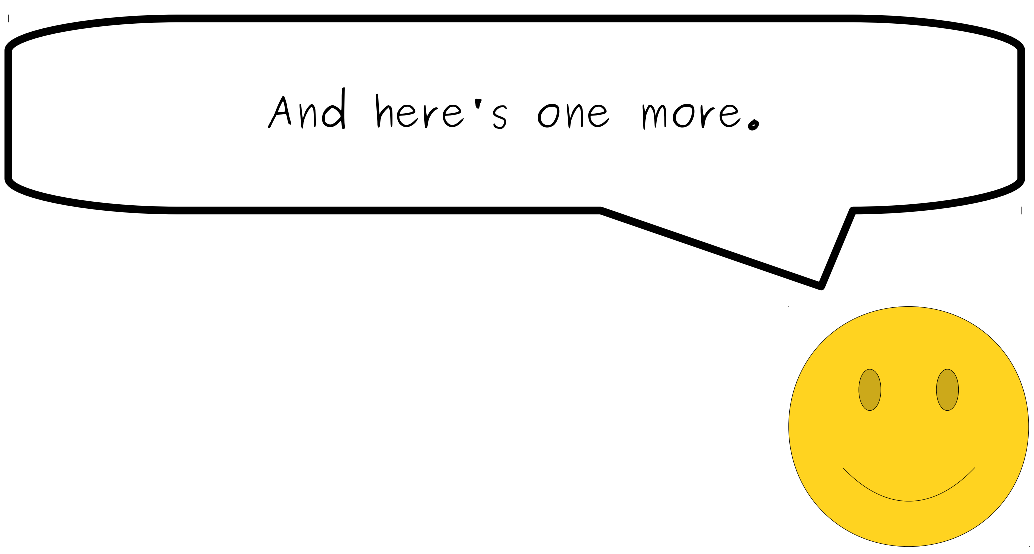

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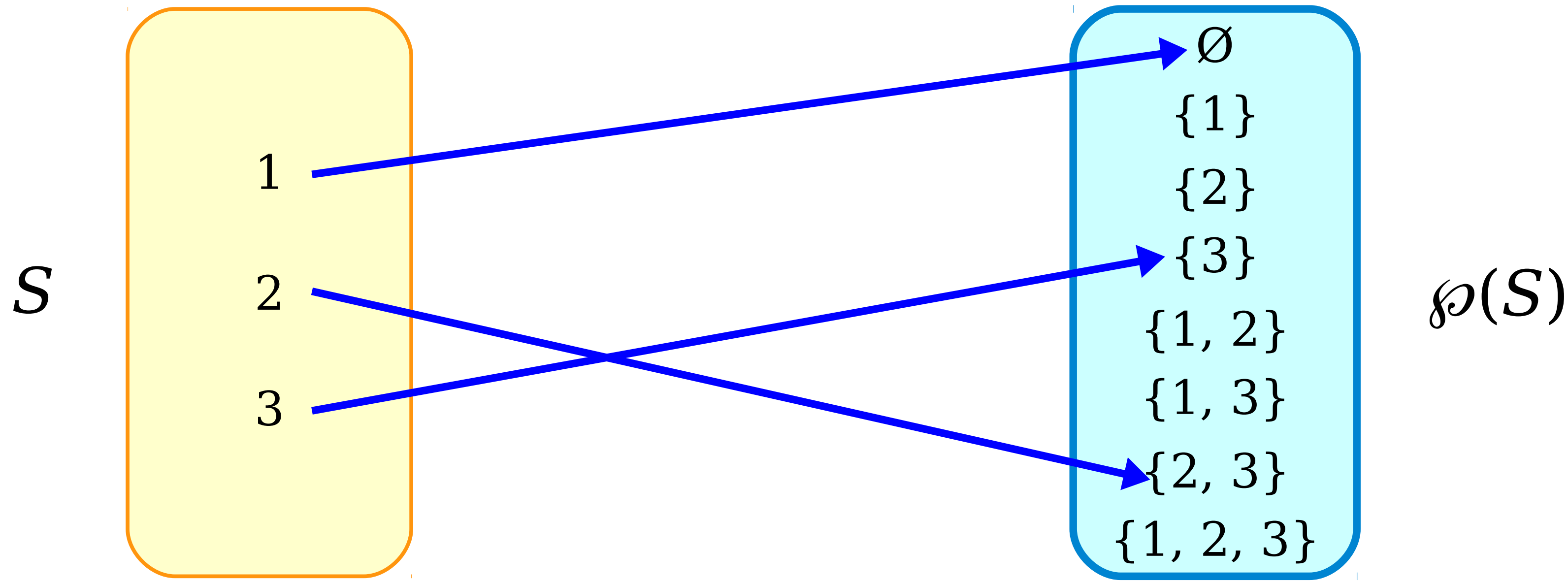

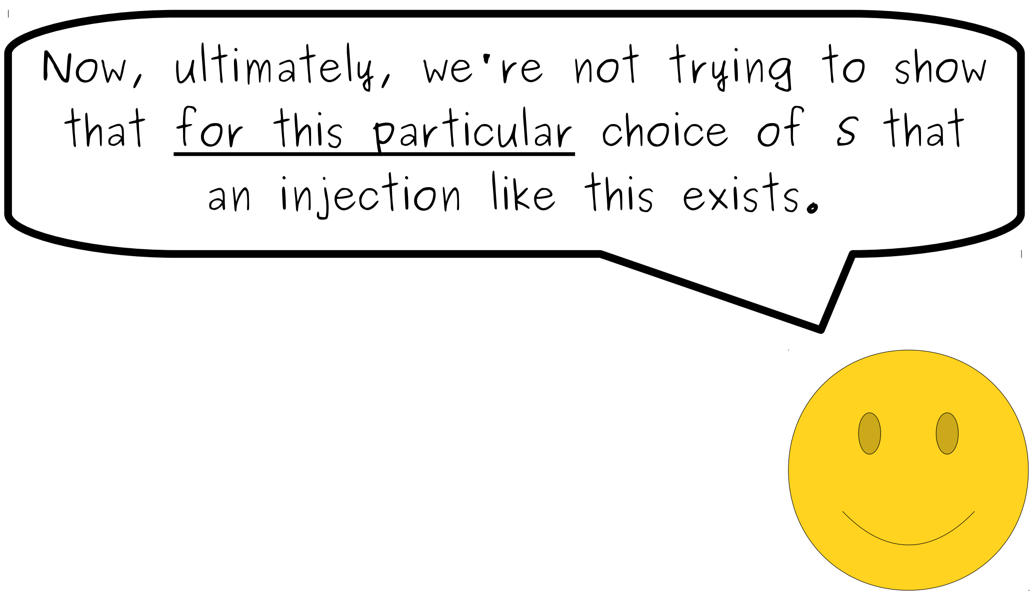

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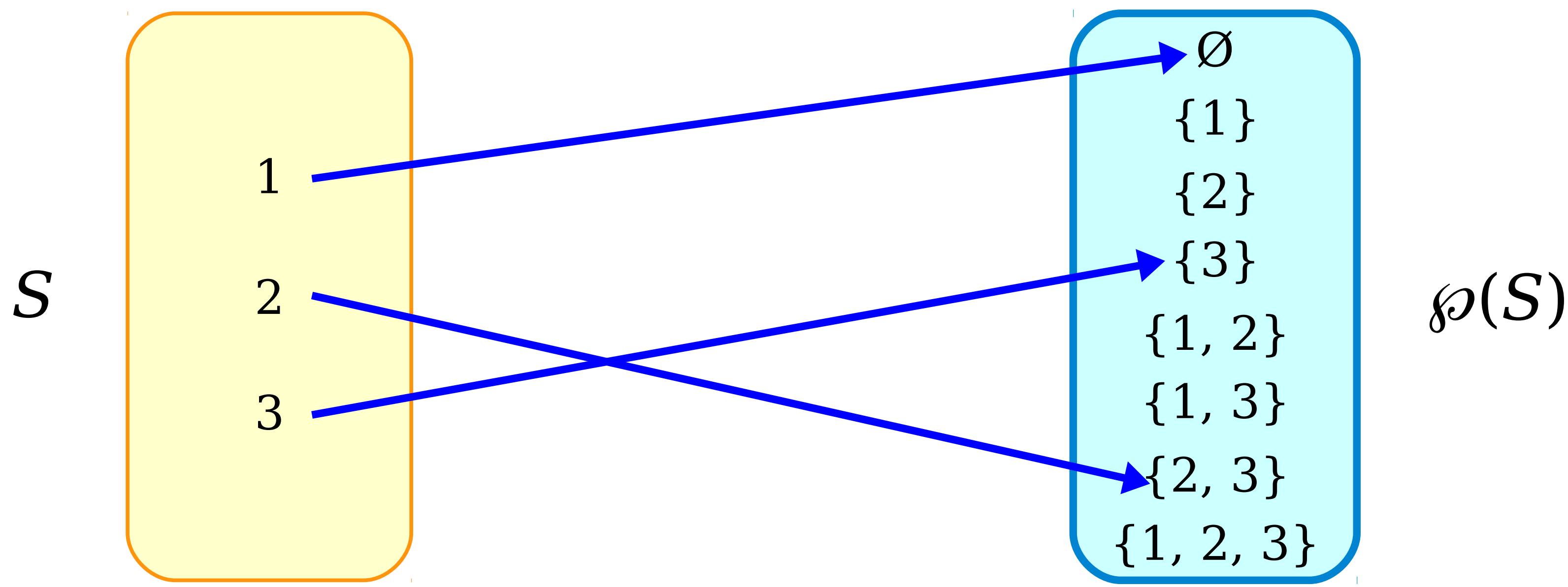

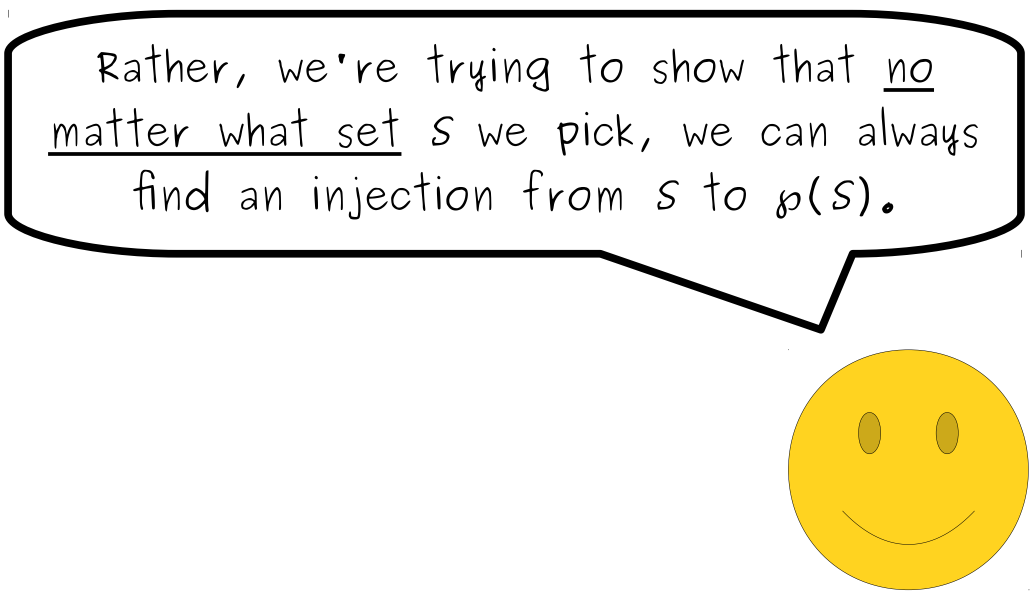

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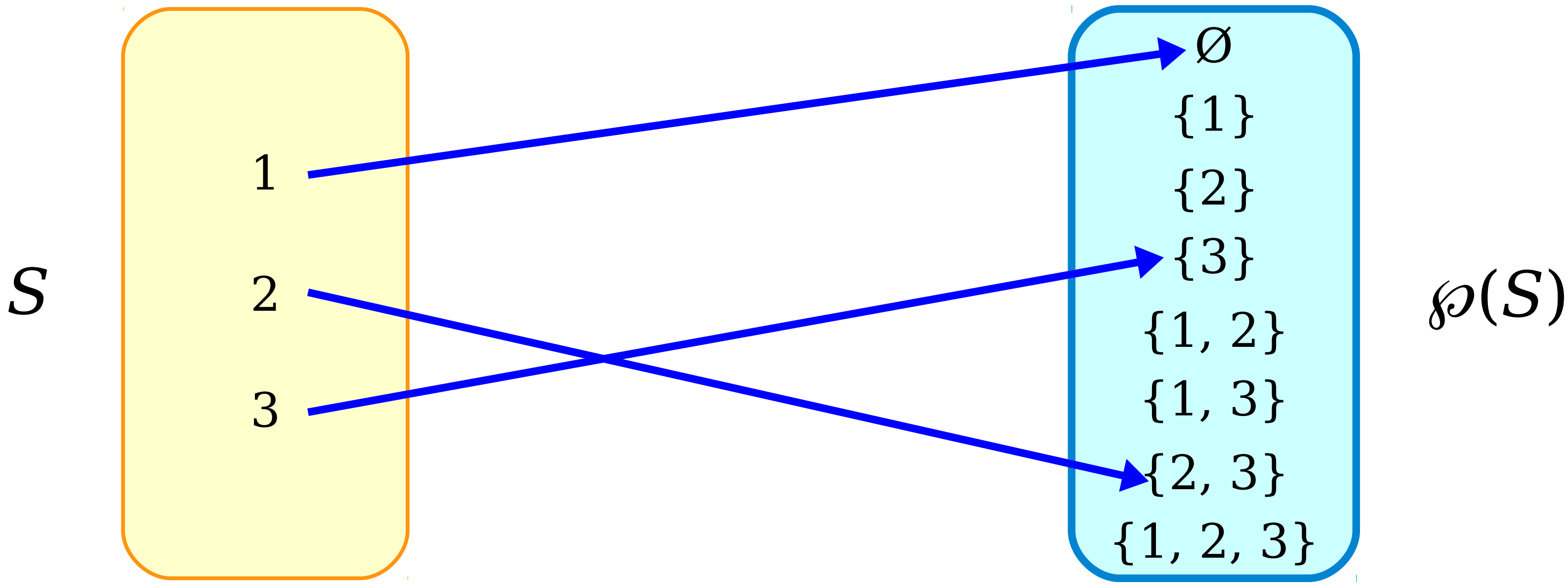

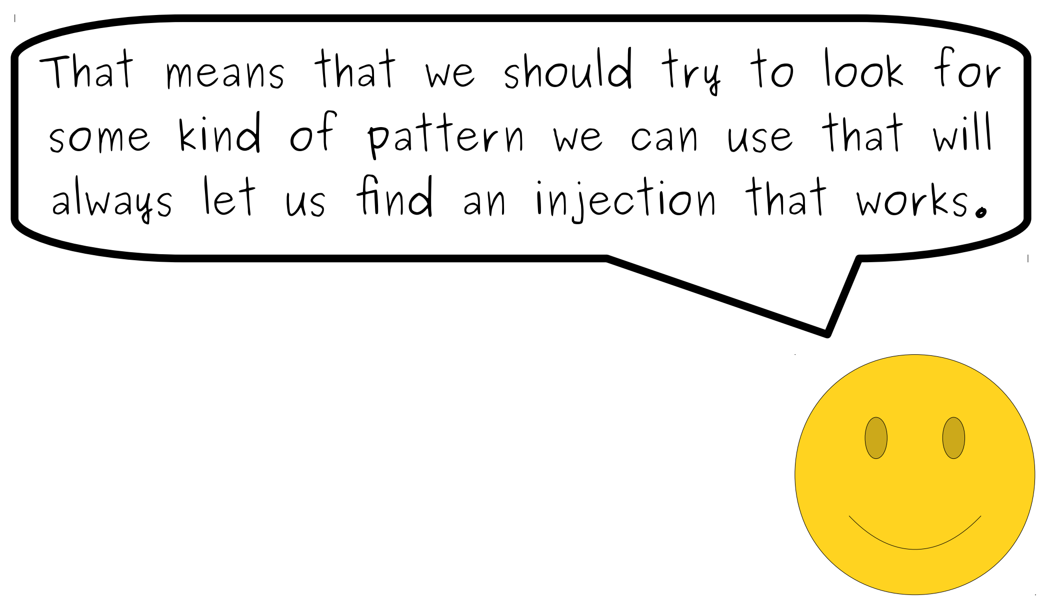

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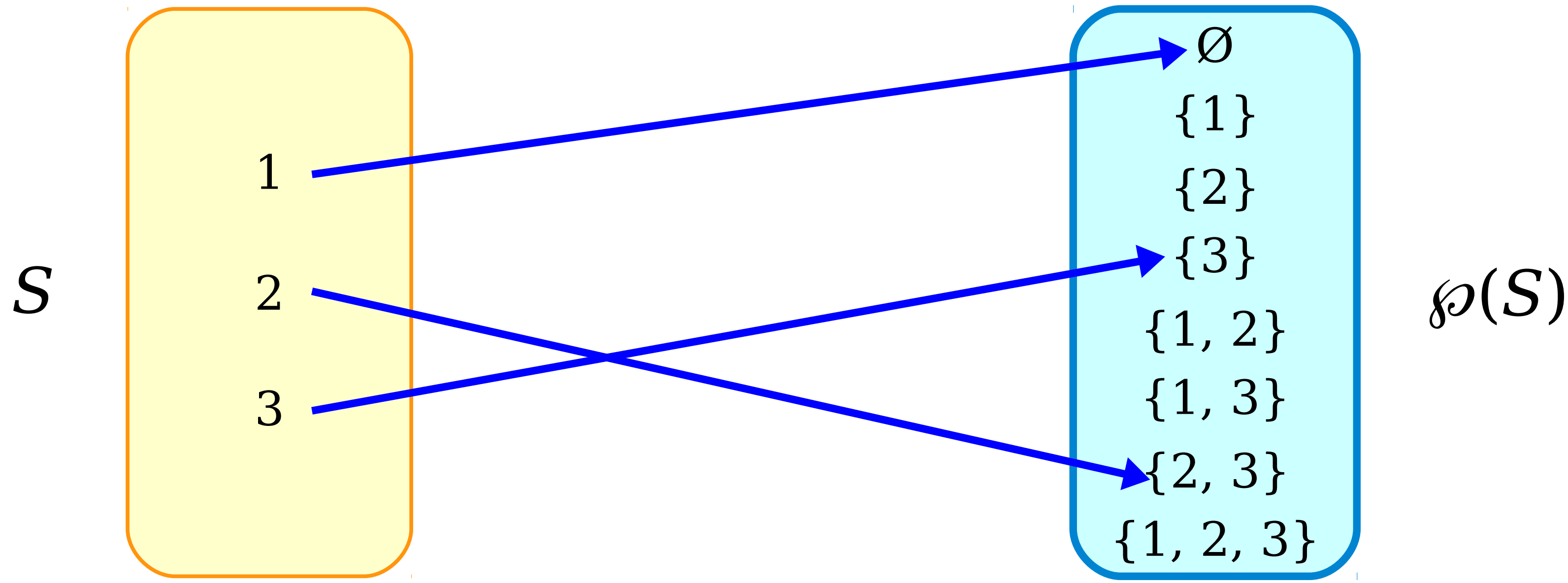

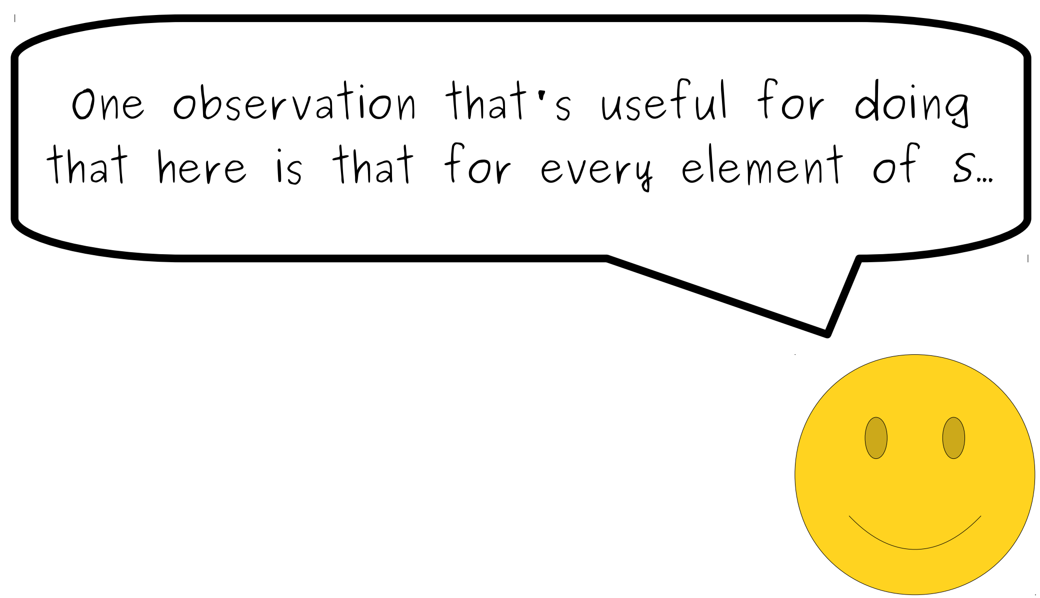

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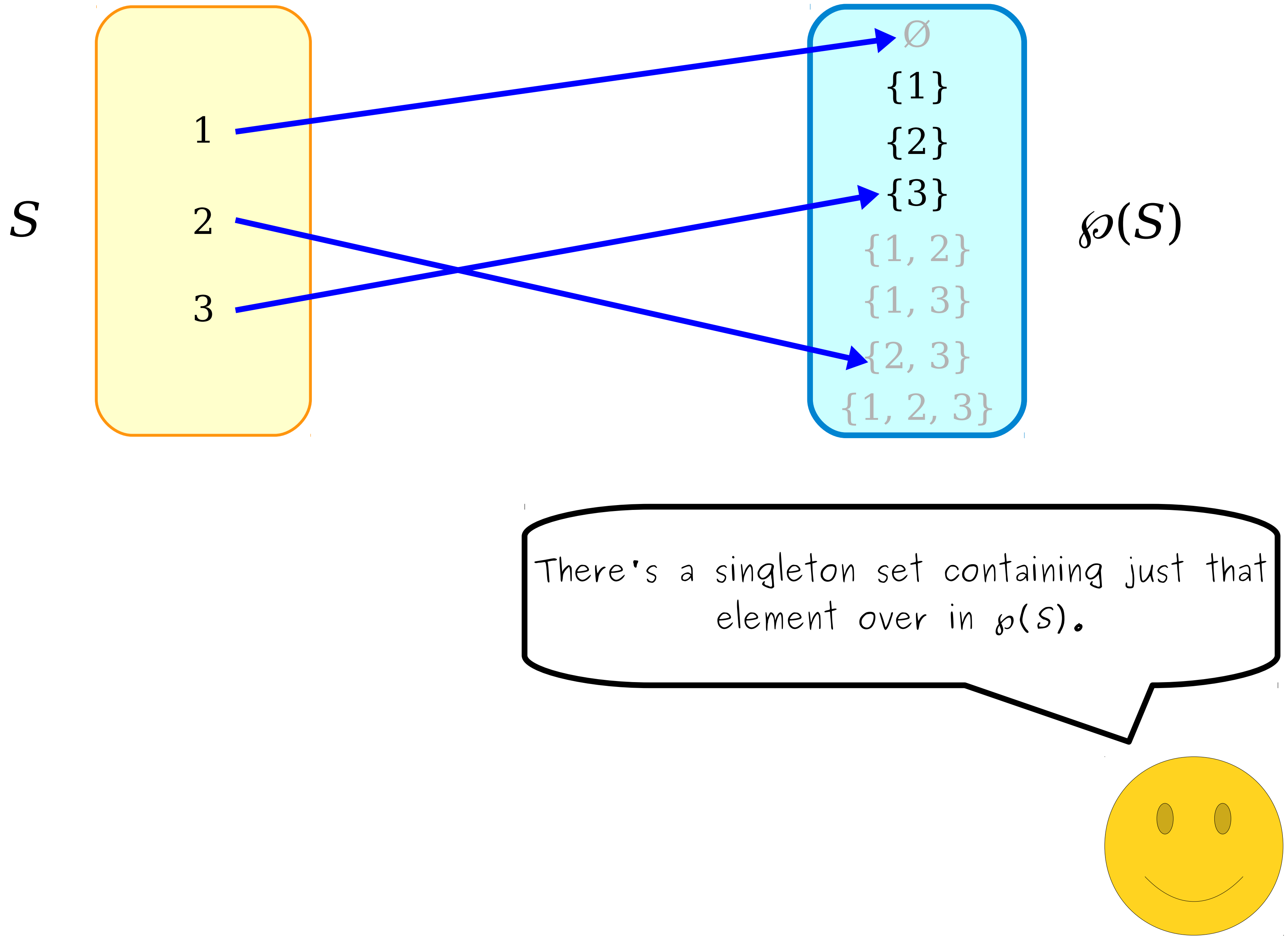

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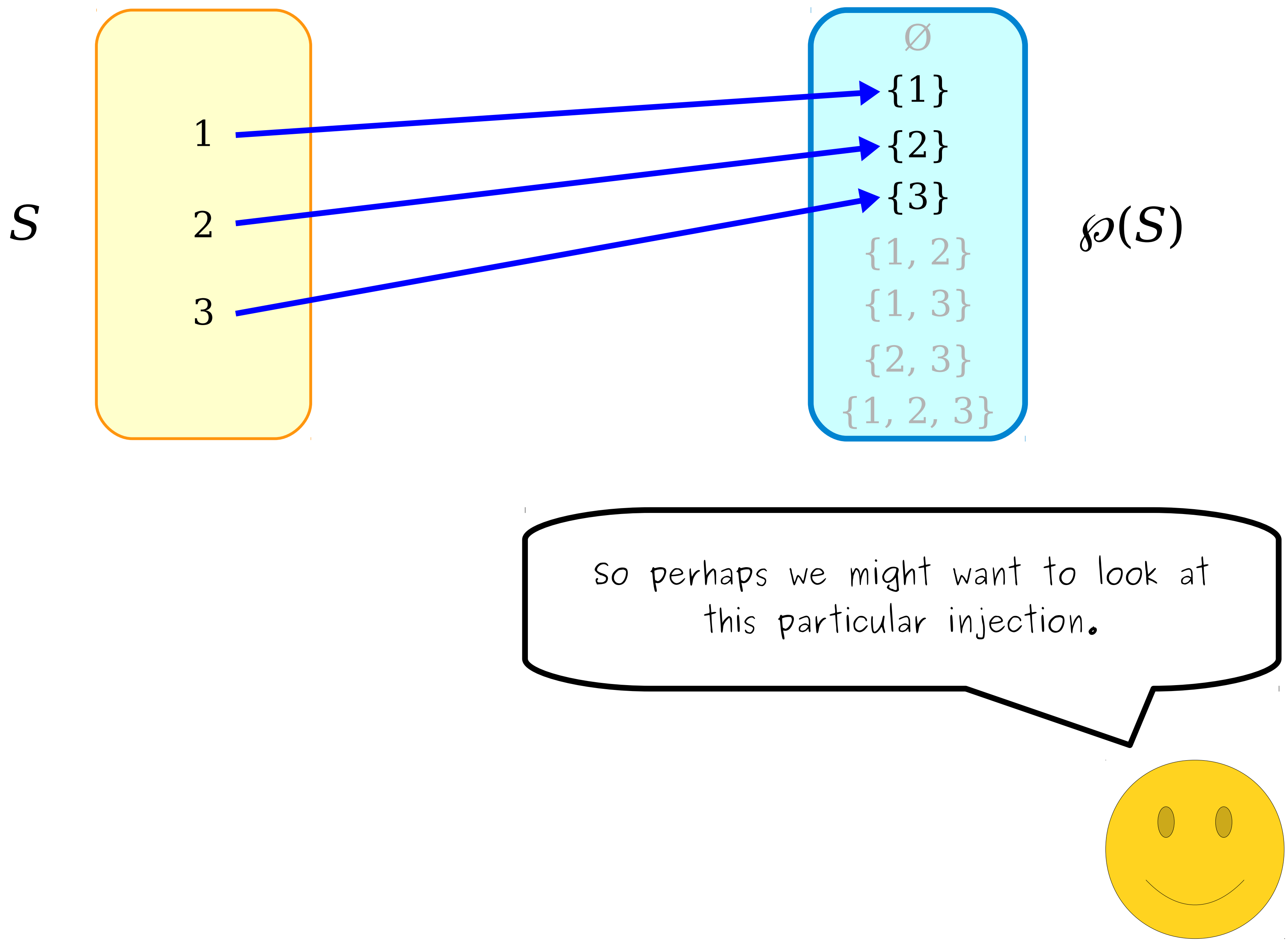

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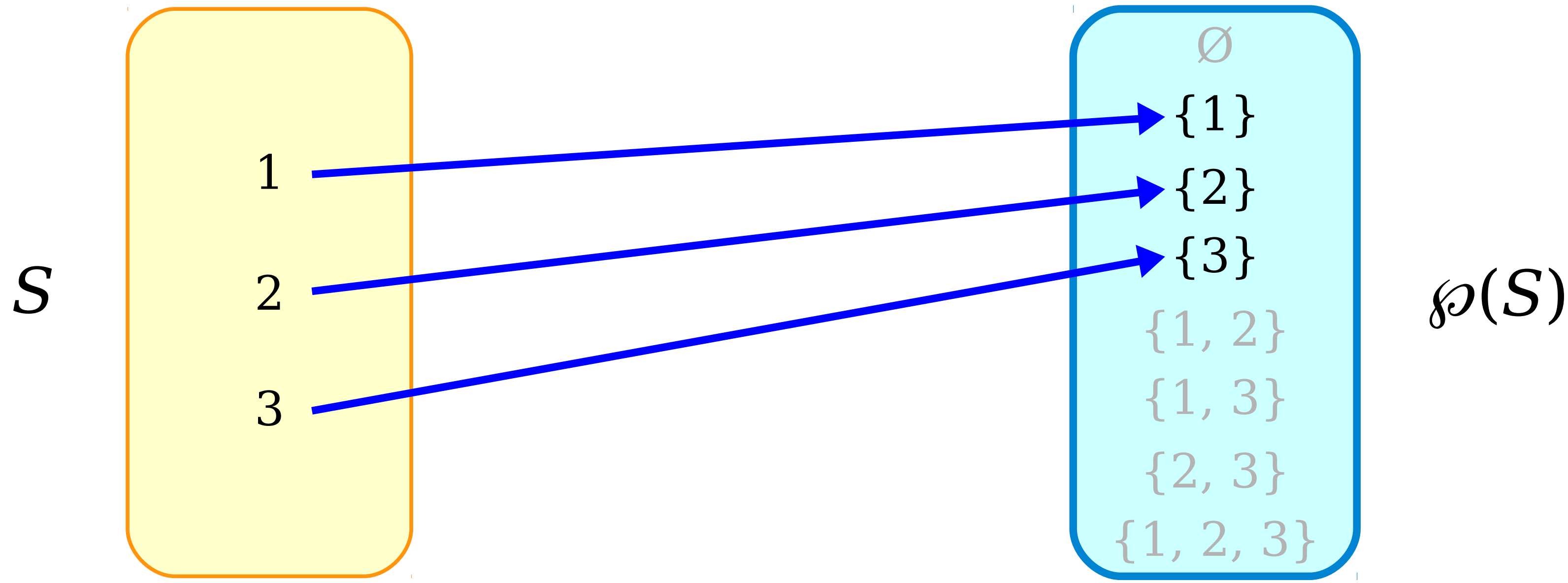
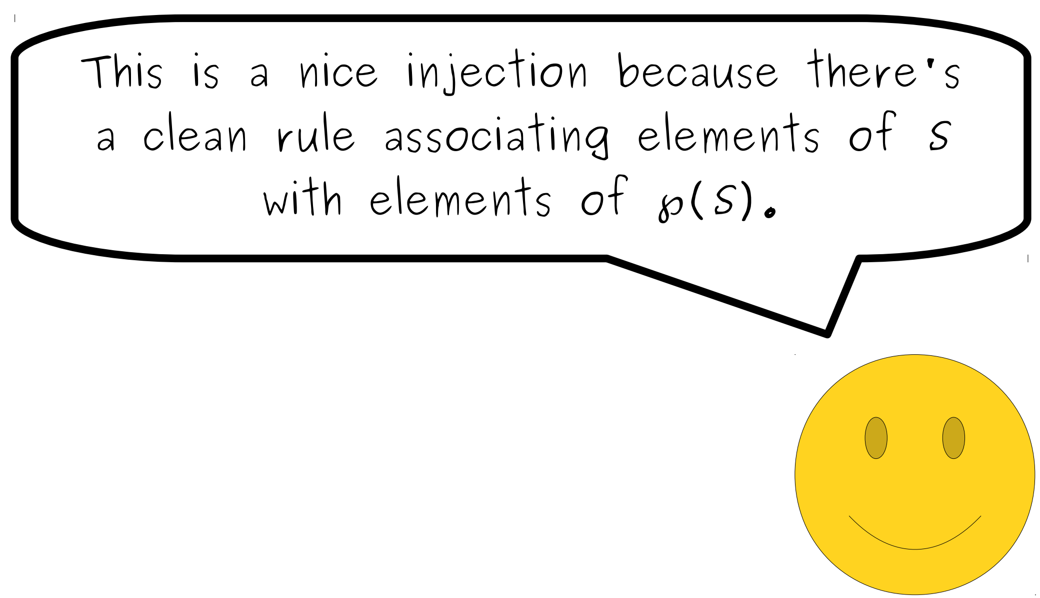


$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$



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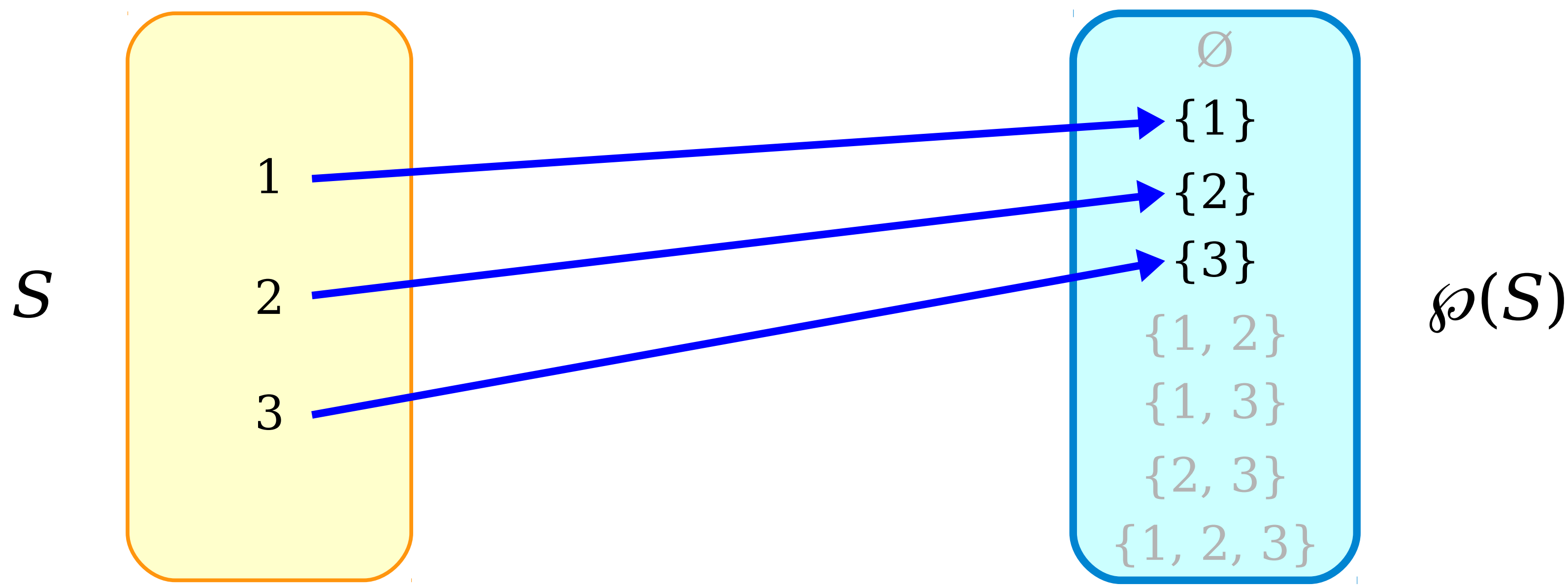
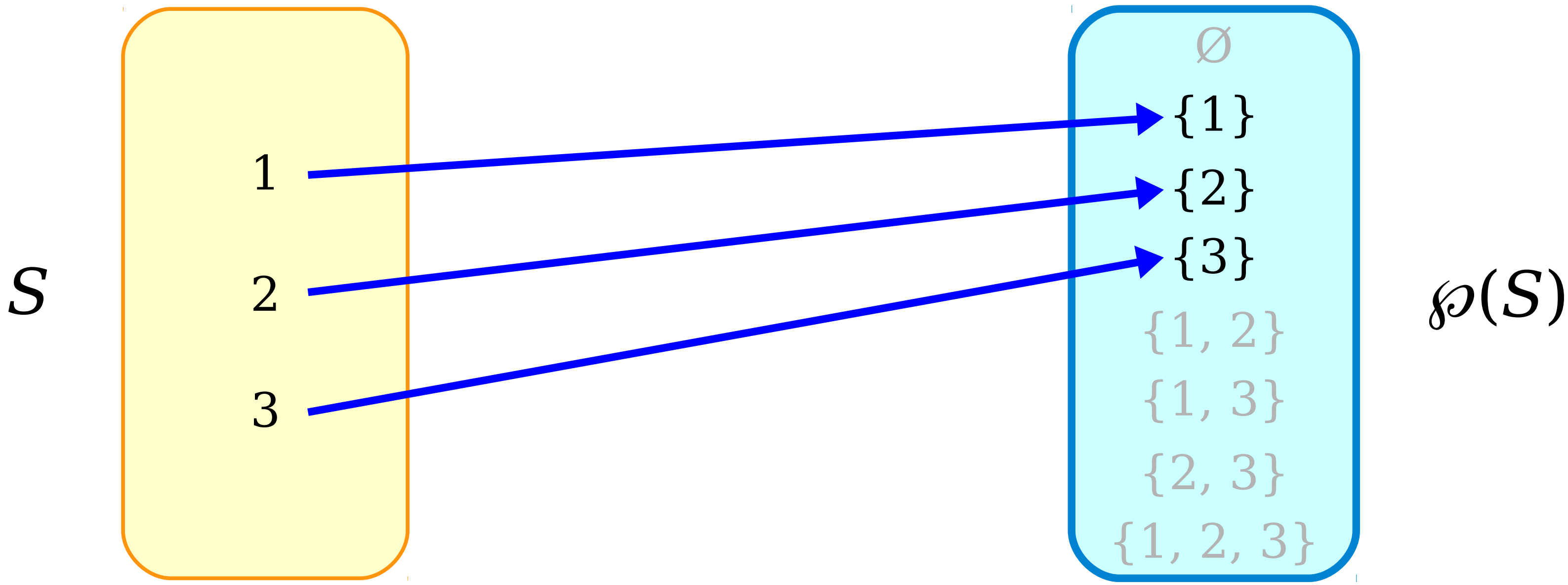


$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$

With a little thought, we can see that no matter what set we choose for S, this function will always be well-defined.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



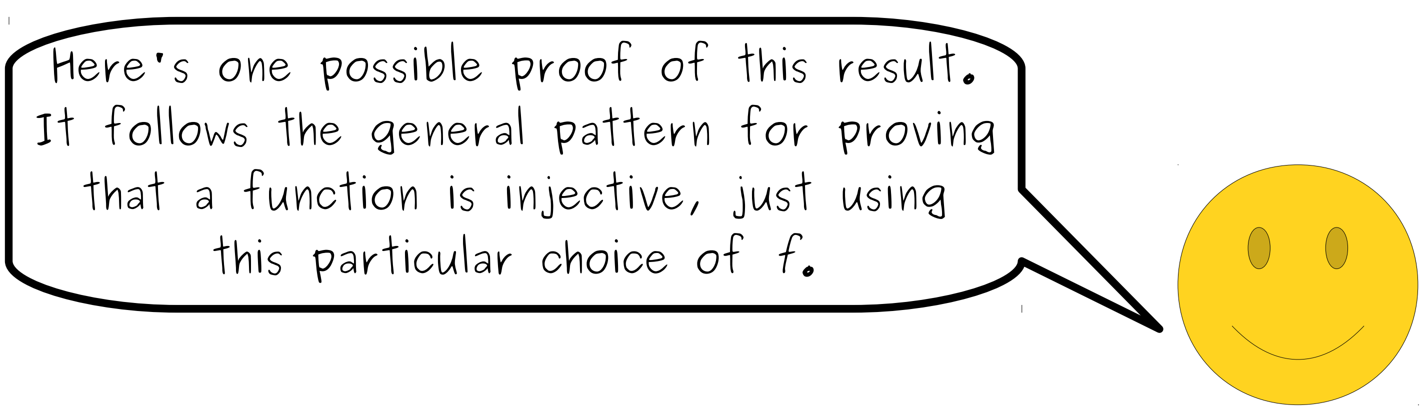
So if we want to prove the above lemma, perhaps we should try showing that this particular function always works as an injection from S to $\wp(S)$.

$$\boldsymbol{f : S \to \wp(S)}$$
$$\boldsymbol{f(x) = \{x\}}$$

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

Proof: Let S be any set and consider the function $f : S \to \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.

Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required. ■



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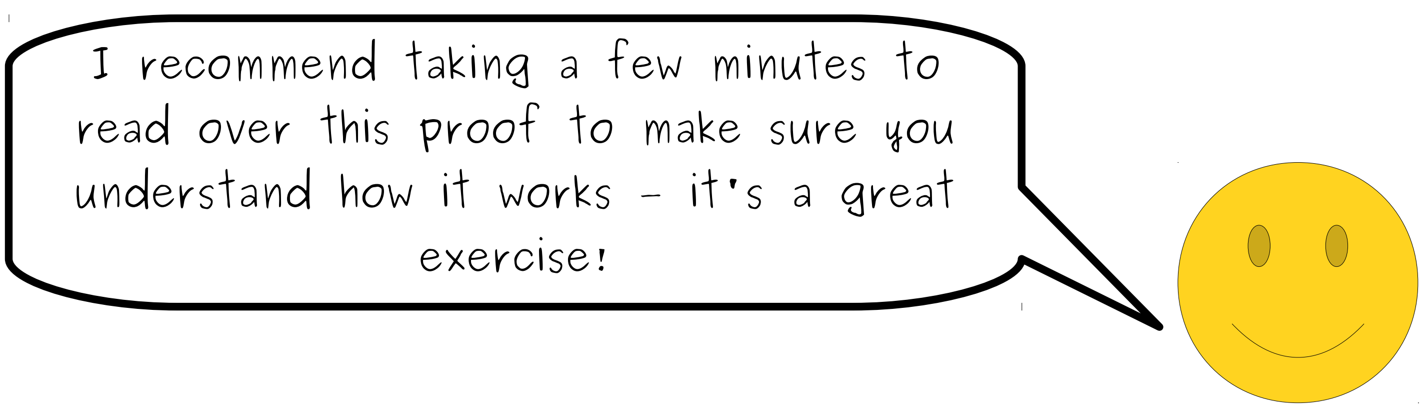
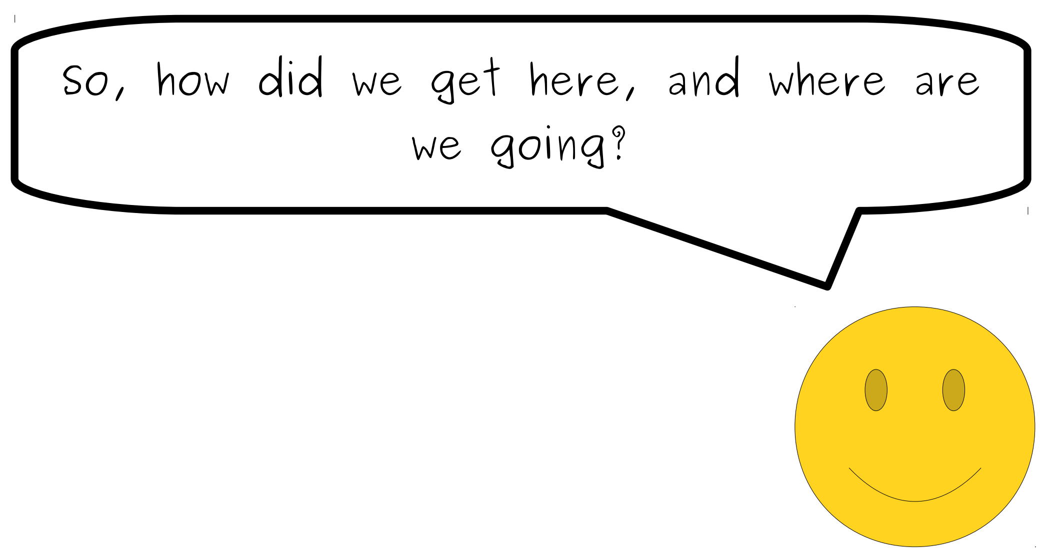
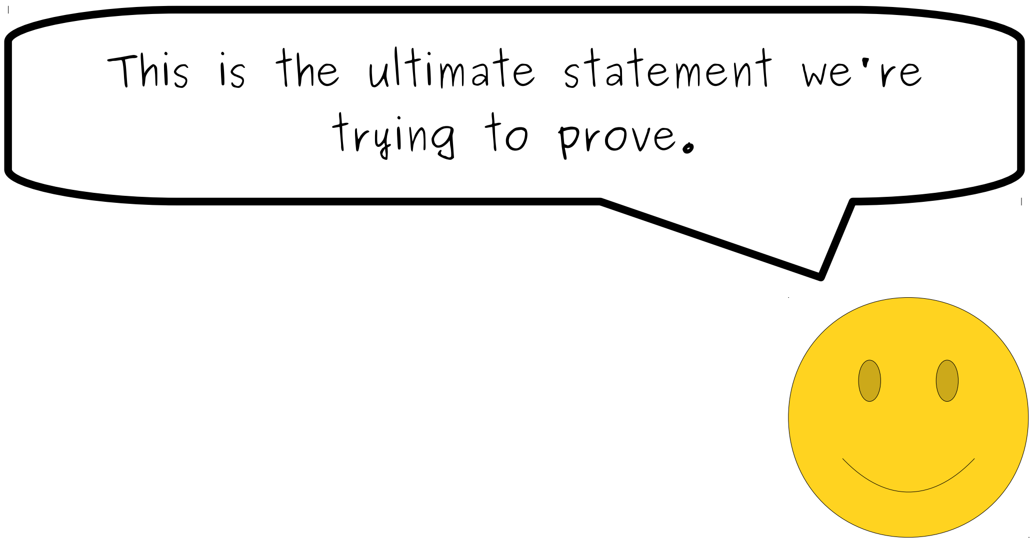



So, how did we get here, and where are we going?

If S is a set, then $|S| < |\wp(S)|$



There *is* an injection $f : S \to \wp(S)$
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

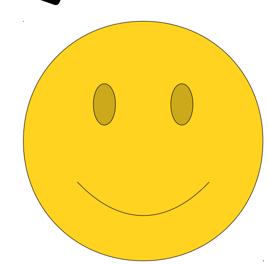
As we saw earlier, this means that we need to prove these two statements.

There *is* an injection $f : S \to \wp(S)$ ✓

There is *no* bijection $f : S \to \wp(S)$

If *S* is a set, then $|S| < |\wp(S)|$

We just finished that first step – great!

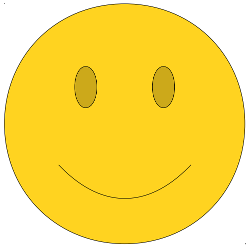


There *is* an injection $f : S \to \wp(S)$ ✓

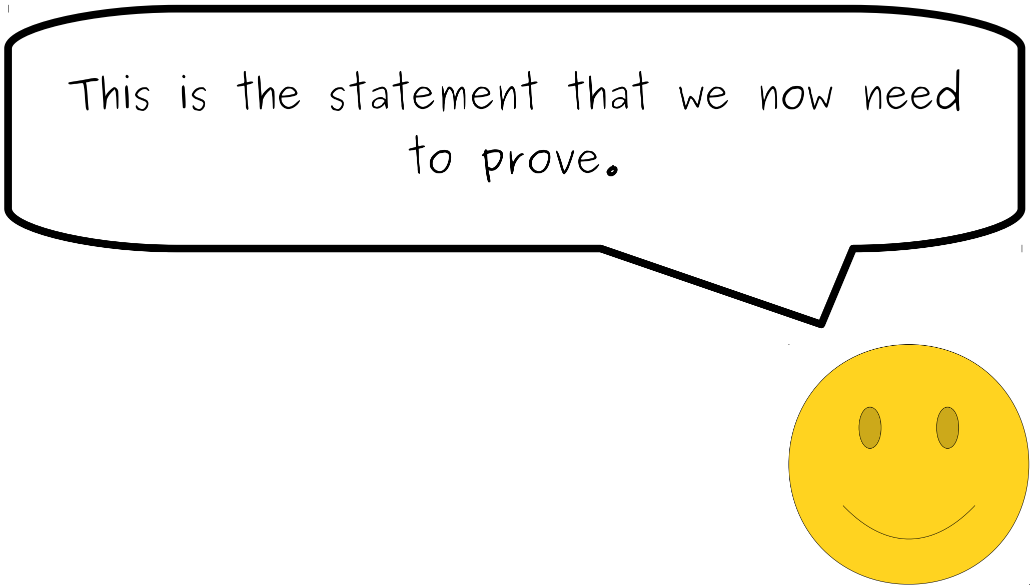
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

Now, we need to prove this part… and that's where we're going to use a diagonal argument.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



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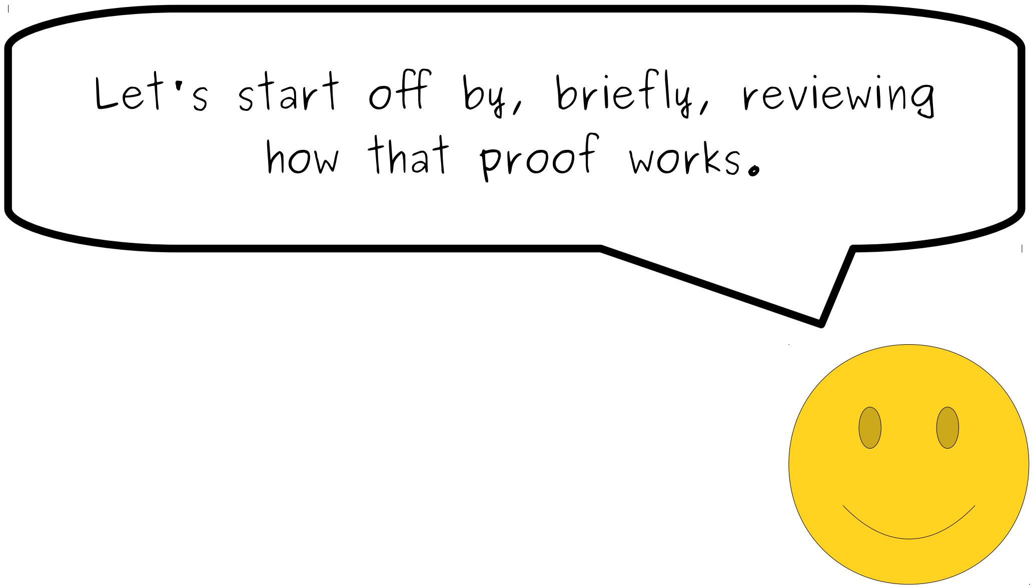


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0

x_1

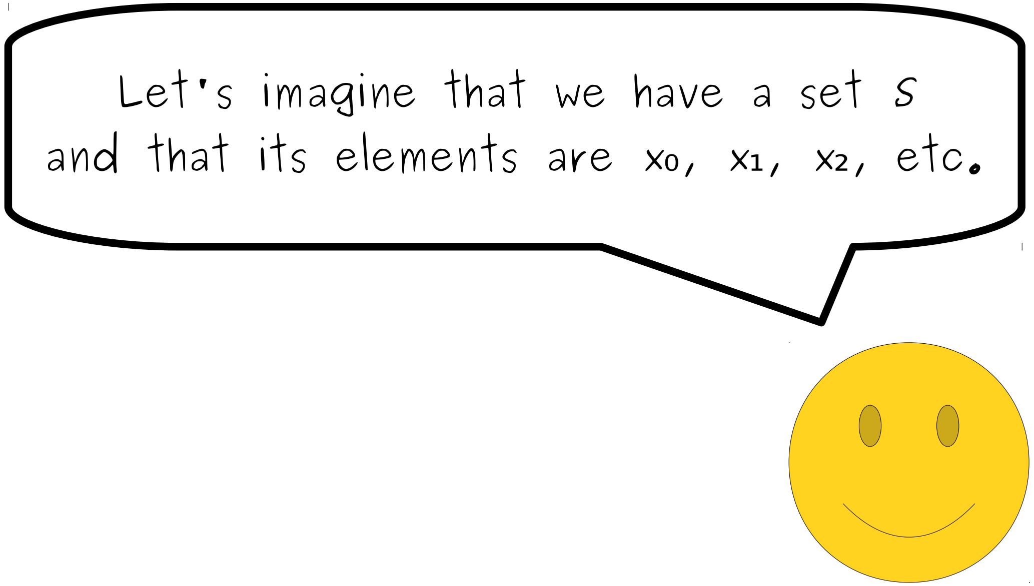
x_2

x_3

x_4

x_5

$\dots$



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x_0

x_1

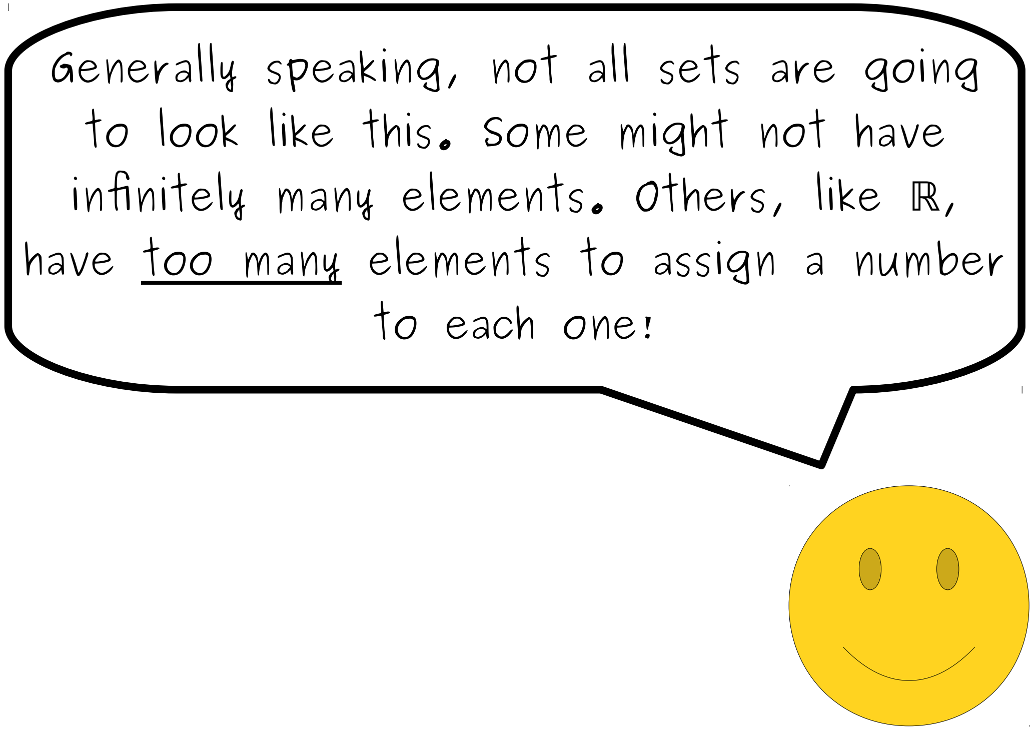
x_2

x_3

x_4

x_5

$\dots$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0

x_1

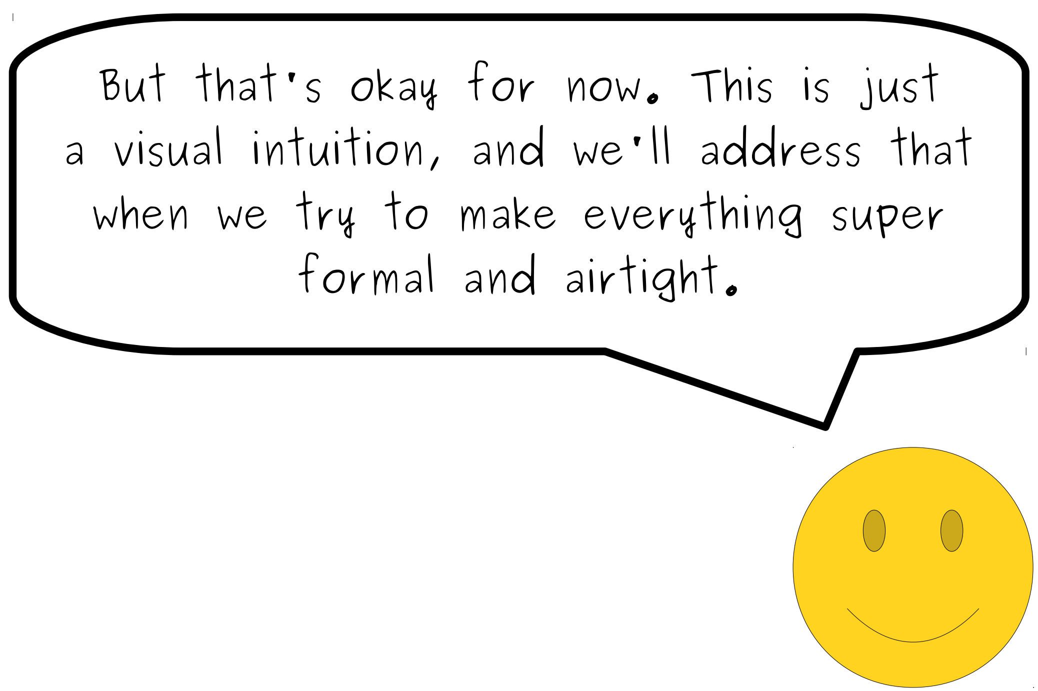
x_2

x_3

x_4

x_5

…



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x_0

x_1

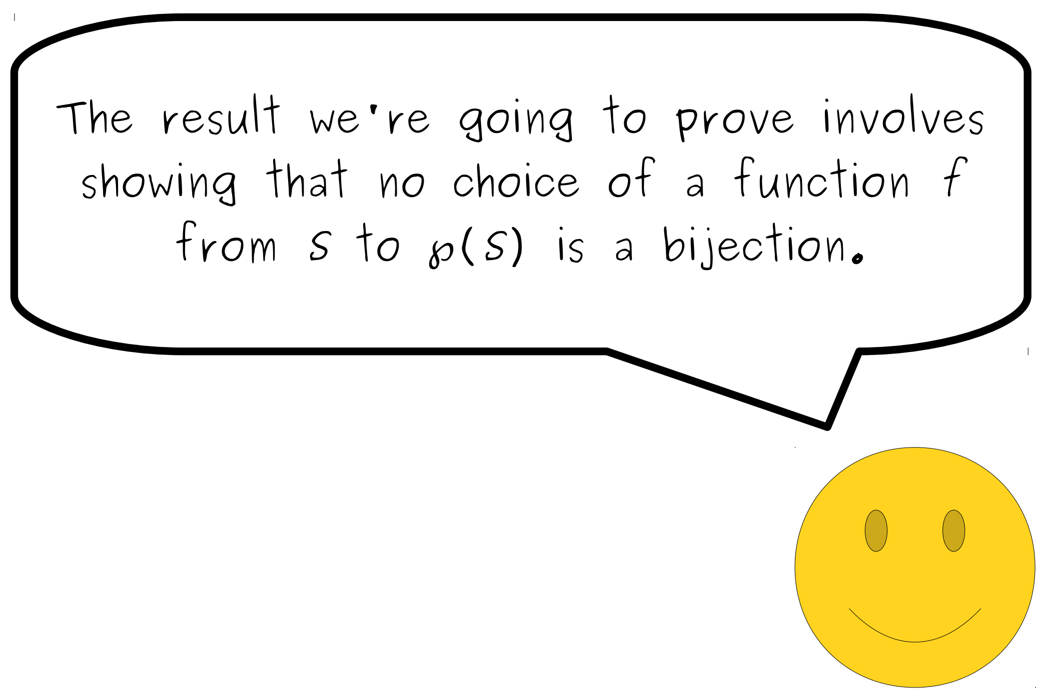
x_2

x_3

x_4

x_5

$\dots$



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x_0

x_1

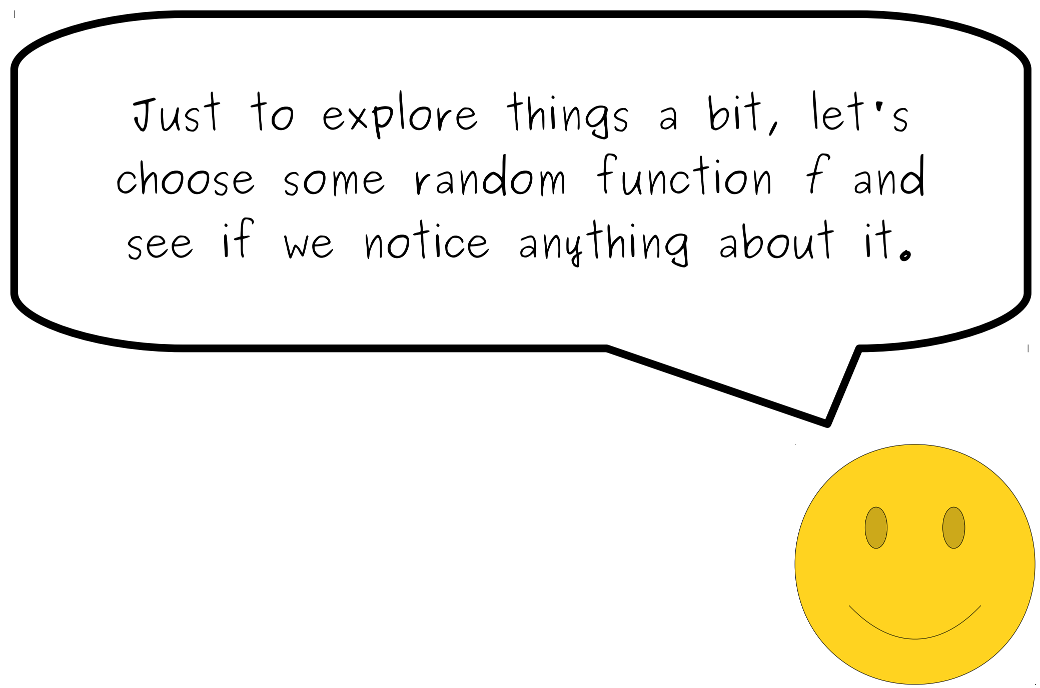
x_2

x_3

x_4

x_5

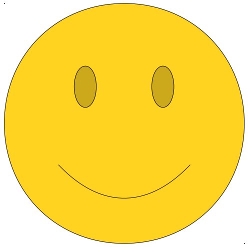
$\dots$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

$$
\begin{array}{lcl}
x_0 & \longrightarrow & \{ x_0, x_2, x_4, \dots \} \\
x_1 & \longrightarrow & \{ x_0, x_3, x_4, \dots \} \\
x_2 & \longrightarrow & \{ x_4, \dots \} \\
x_3 & \longrightarrow & \{ x_1, x_3, x_4, \dots \} \\
x_4 & \longrightarrow & \{ x_0, x_5, \dots \} \\
x_5 & \longrightarrow & \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \} \\
\dots & &
\end{array}
$$

Here's one example function f. There's no particular pattern here in the function we picked – it's just a random function to help us build an intuition.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

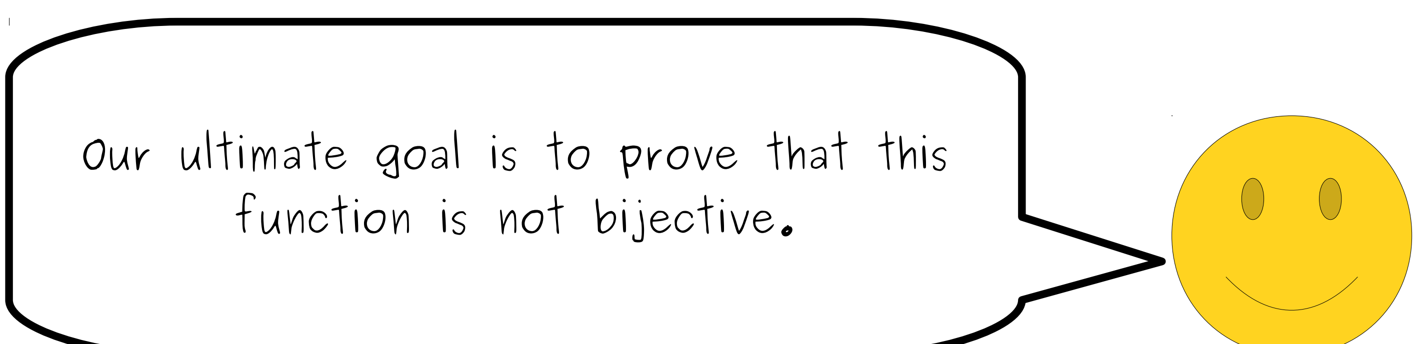
$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

$$\dots$$



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$$x_1 \longrightarrow \{\, x_0, x_3, x_4, \dots \,\}$$

$$x_2 \longrightarrow \{\, x_4, \dots \,\}$$

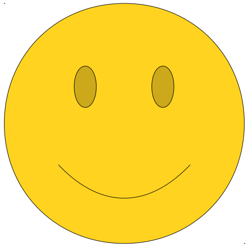
$$x_3 \longrightarrow \{\, x_1, x_3, x_4, \dots \,\}$$

$$x_4 \longrightarrow \{\, x_0, x_5, \dots \,\}$$

$$x_5 \longrightarrow \{\, x_0, x_1, x_2, x_3, x_4, x_5, \dots \,\}$$

$$\dots$$

That's going to be a challenge, since we can't assume that this function follows any nice pattern.



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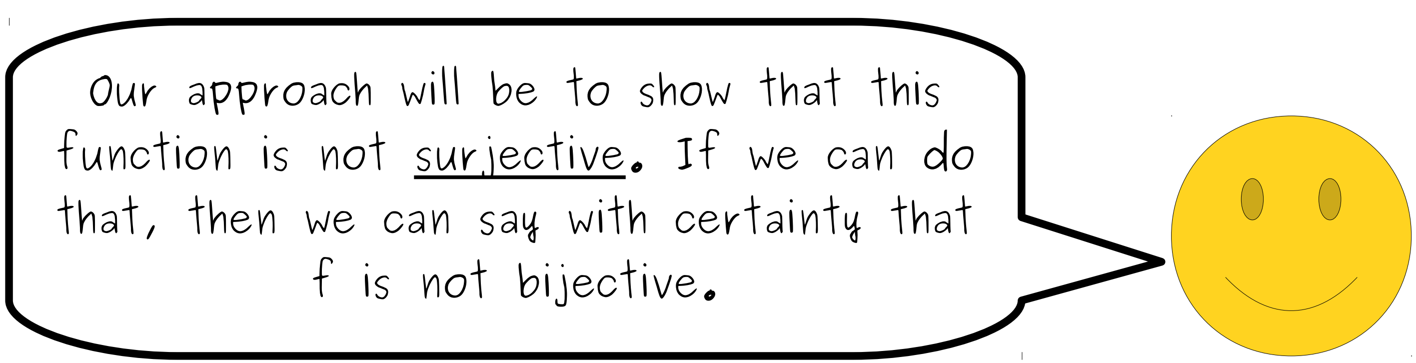
$x_2 \longrightarrow \{ x_4, \ldots \}$

$x_3 \longrightarrow \{ x_1, x_3, x_4, \ldots \}$

$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

$\ldots$



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$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

$$\dots$$

To do so, we're going to have to find a way to show that there's some subset of S that isn't covered by our function.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

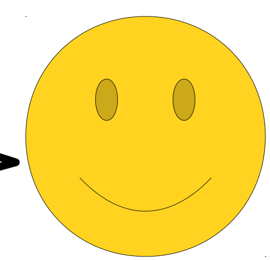
$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

$$\dots$$

Visually, that means that we need to find a set that's not paired with anything in the left-hand column.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

$x_0 \longrightarrow \{\ x_0, x_2, x_4, \dots\ \}$

$x_1 \longrightarrow \{\ x_0, x_3, x_4, \dots\ \}$

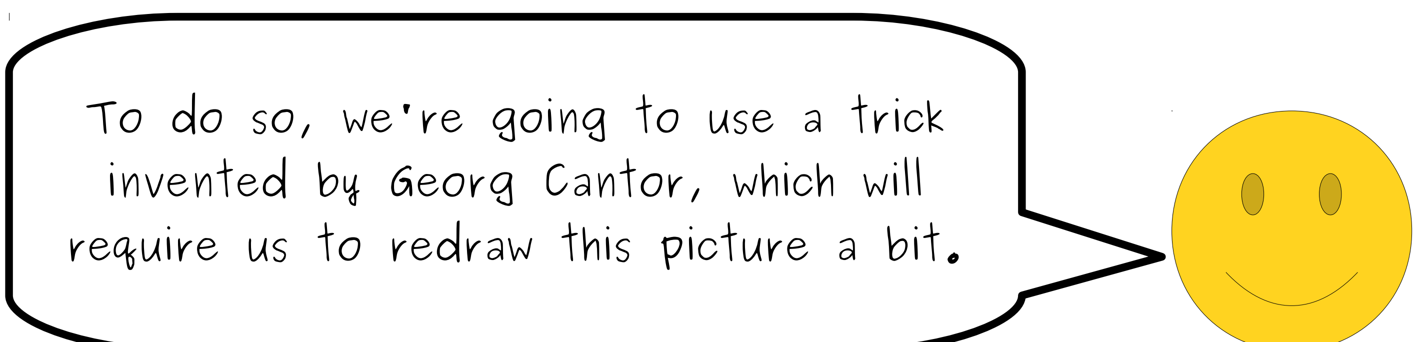
$x_2 \longrightarrow \{\ x_4, \dots\ \}$

$x_3 \longrightarrow \{\ x_1, x_3, x_4, \dots\ \}$

$x_4 \longrightarrow \{\ x_0, x_5, \dots\ \}$

$x_5 \longrightarrow \{\ x_0, x_1, x_2, x_3, x_4, x_5, \dots\ \}$

$\dots$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$

$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$

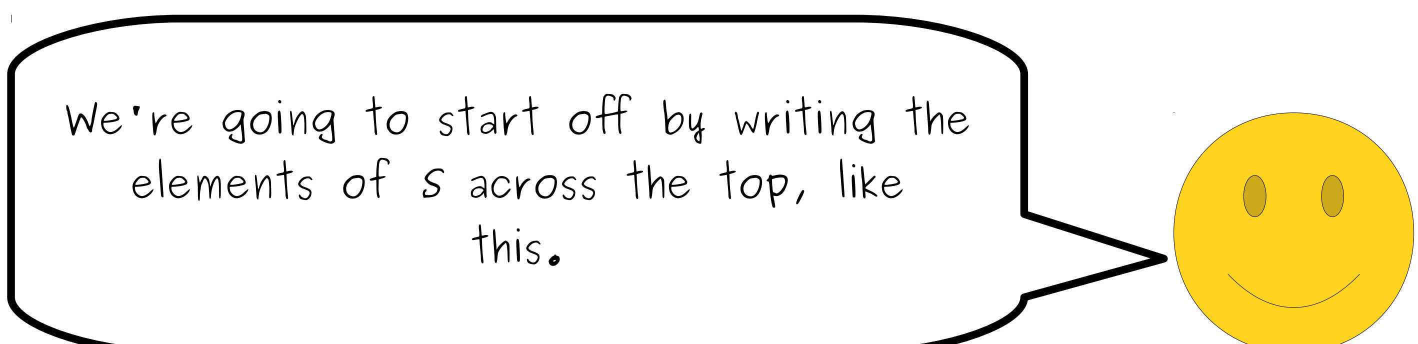
$x_2 \longrightarrow \{ x_4, \dots \}$

$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$

$x_4 \longrightarrow \{ x_0, x_5, \dots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

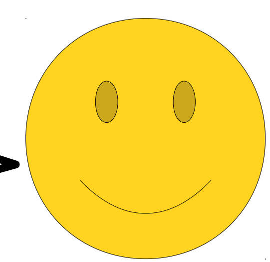
…



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	...
$x_0 \longrightarrow$	{ x_0,		x_2,		x_4,		... }
$x_1 \longrightarrow$	{ x_0,			x_3,	x_4,		... }
$x_2 \longrightarrow$	{				x_4,		... }
$x_3 \longrightarrow$	{	x_1,		x_3,	x_4,		... }
$x_4 \longrightarrow$	{ x_0,					x_5,	... }
$x_5 \longrightarrow$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	... }
...							

We can then imagine spacing out the elements that are in each of these sets so that we start to have something that looks more like a grid.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	$\longrightarrow$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
x_1	$\longrightarrow$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
x_2	$\longrightarrow$	$\{$				$x_4,$		$\dots \}$
x_3	$\longrightarrow$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
x_4	$\longrightarrow$	$\{ x_0,$					$x_5,$	$\dots \}$
x_5	$\longrightarrow$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$

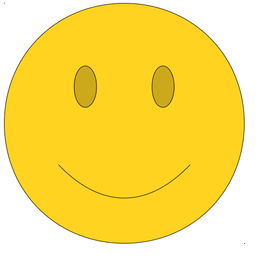
…

Cantor's insight – which is the real trick behind the proof – is the following.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
x_0	$\to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
x_1	$\to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
x_2	$\to$	$\{$				$x_4,$		$\dots \}$
x_3	$\to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
x_4	$\to$	$\{ x_0,$					$x_5,$	$\dots \}$
x_5	$\to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$								

The trick, as you saw in lecture one, is to look at this main diagonal and use it to build a set that's not mapped to.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \to \{ x_0, x_2, x_4, \ldots \}$

$x_1 \to \{ x_0, x_3, x_4, \ldots \}$

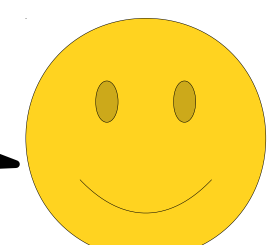
$x_2 \to \{ x_4, \ldots \}$

$x_3 \to \{ x_1, x_3, x_4, \ldots \}$

$x_4 \to \{ x_0, x_5, \ldots \}$

$x_5 \to \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

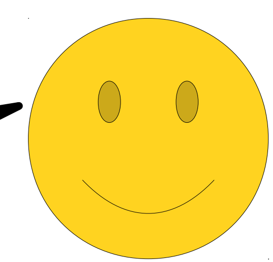
	x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$	$\{ x_0,$			$x_3,$		x_5	$\dots \}$

First, we're going to form a set of all the elements on the diagonal.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

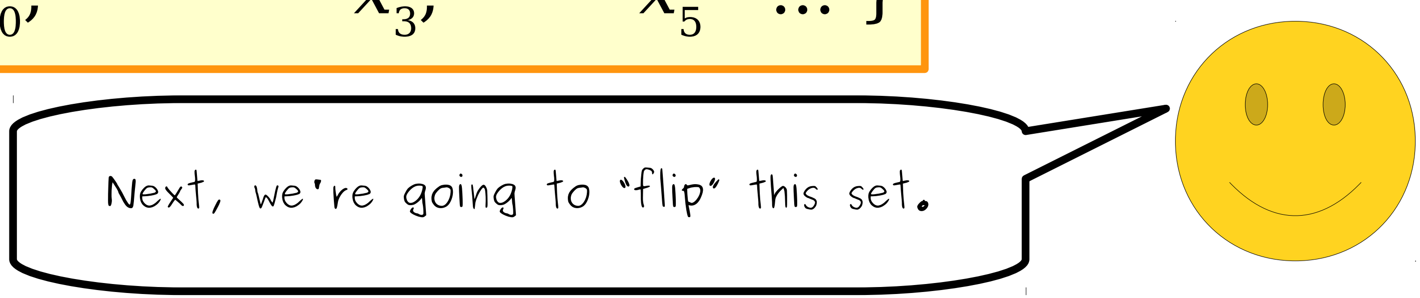
		x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
x_0	$\to$	$\{\ x_0,$		$x_2,$		$x_4,$		$\dots\ \}$
x_1	$\to$	$\{\ x_0,$			$x_3,$	$x_4,$		$\dots\ \}$
x_2	$\to$	$\{$				$x_4,$		$\dots\ \}$
x_3	$\to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots\ \}$
x_4	$\to$	$\{\ x_0,$					$x_5,$	$\dots\ \}$
x_5	$\to$	$\{\ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots\ \}$
$\dots$		$\{\ x_0,$			$x_3,$		x_5	$\dots\ \}$

This color-coding hopefully makes it a bit easier to see where everything comes from.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	$\to$	{ x_0,		x_2,		x_4,		… }
x_1	$\to$	{ x_0,			x_3,	x_4,		… }
x_2	$\to$	{				x_4,		… }
x_3	$\to$	{	x_1,		x_3,	x_4,		… }
x_4	$\to$	{ x_0,					x_5,	… }
x_5	$\to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…		$\boxed{\{\ x_0,}$			$\boxed{x_3,}$		$\boxed{x_5}$	$\boxed{\ldots\ \}}$

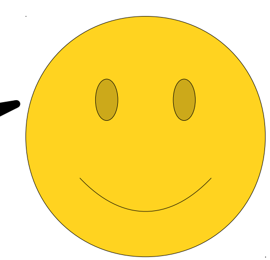


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\to$ {	x_0,		x_2,		x_4,		…	}
x_1	$\to$ {	x_0,			x_3,	x_4,		…	}
x_2	$\to$ {					x_4,		…	}
x_3	$\to$ {		x_1,		x_3,	x_4,		…	}
x_4	$\to$ {	x_0,					x_5,	…	}
x_5	$\to$ {	x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…									

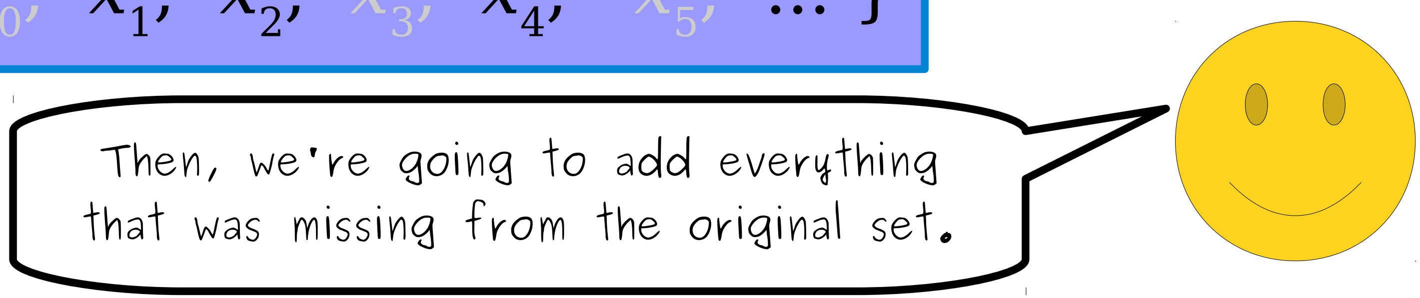
$\{ x_0, \quad x_3, \quad x_5 \quad \ldots \}$

We're going to remove everything that used to be in this set…



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	{ x_0,		x_2,		x_4,		… }
$x_1 \to$	{ x_0,			x_3,	x_4,		… }
$x_2 \to$	{				x_4,		… }
$x_3 \to$	{	x_1,		x_3,	x_4,		… }
$x_4 \to$	{ x_0,					x_5,	… }
$x_5 \to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, x_2, x_4, \ldots \}$

$x_1 \longrightarrow \{ x_0, x_3, x_4, \ldots \}$

$x_2 \longrightarrow \{ x_4, \ldots \}$

$x_3 \longrightarrow \{ x_1, x_3, x_4, \ldots \}$

$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

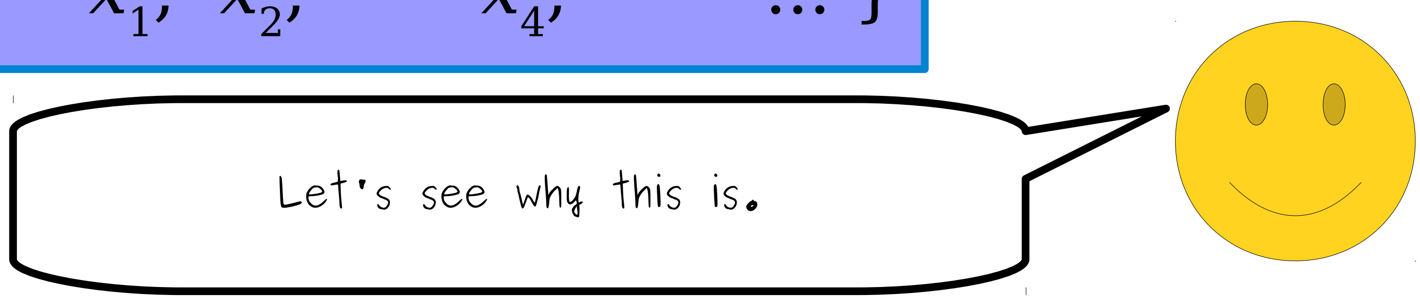
$\ldots$

$\boxed{\{ x_1, x_2, x_4, \ldots \}}$

This new set – which is called the <u>diagonal set</u> – is specifically built to be different from every row above.

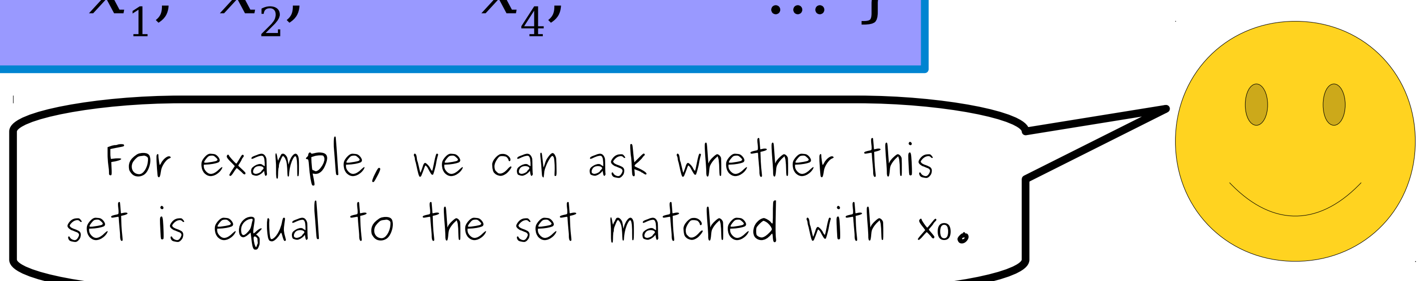
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longrightarrow$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \longrightarrow$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \longrightarrow$	$\{$				$x_4,$		$\dots \}$
$x_3 \longrightarrow$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \longrightarrow$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \longrightarrow$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$							
	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

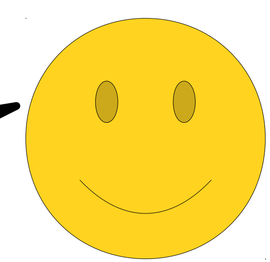
		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\longrightarrow$	{ x_0,		x_2,		x_4,		…	}
x_1	$\longrightarrow$	{ x_0,			x_3,	x_4,		…	}
x_2	$\longrightarrow$	{				x_4,		…	}
x_3	$\longrightarrow$	{	x_1,		x_3,	x_4,		…	}
x_4	$\longrightarrow$	{ x_0,					x_5,	…	}
x_5	$\longrightarrow$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…									
		{	x_1,	x_2,		x_4,		…	}



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\to$	$\{\ x_0,$		$x_2,$		$x_4,$		…	$\}$
x_1	$\to$	$\{\ x_0,$			$x_3,$	$x_4,$		…	$\}$
x_2	$\to$	$\{$				$x_4,$		…	$\}$
x_3	$\to$	$\{$	$x_1,$		$x_3,$	$x_4,$		…	$\}$
x_4	$\to$	$\{\ x_0,$					$x_5,$	…	$\}$
x_5	$\to$	$\{\ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	…	$\}$
…									
		$\{$	$x_1,$	$x_2,$		$x_4,$		…	$\}$

But we can see that's not the case. The set paired with x_0 contains x_0, but our diagonal set doesn't contain x_0.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \to \{ x_0, x_2, x_4, \dots \}$

$x_1 \to \{ x_0, x_3, x_4, \dots \}$

$x_2 \to \{ x_4, \dots \}$

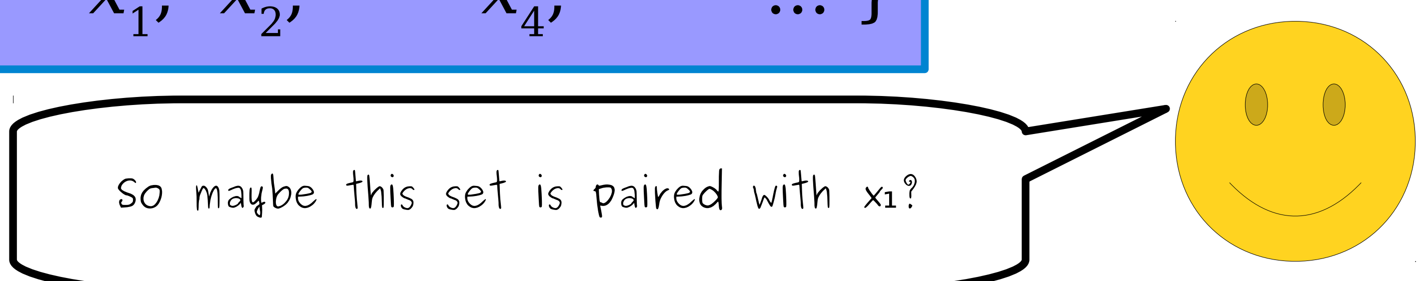
$x_3 \to \{ x_1, x_3, x_4, \dots \}$

$x_4 \to \{ x_0, x_5, \dots \}$

$x_5 \to \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

$\dots$

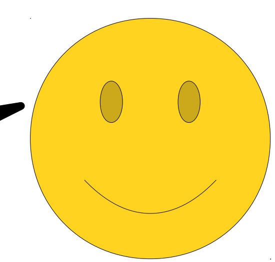
$\boxed{\{ x_1, x_2, x_4, \dots \}}$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

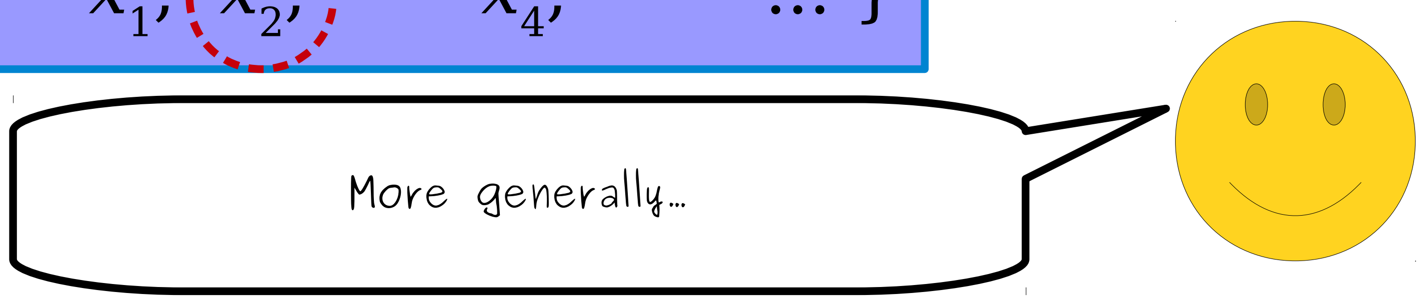
		x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	$\to$	{ x_0,		x_2,		x_4,		… }
x_1	$\to$	{ x_0,			x_3,	x_4,		… }
x_2	$\to$	{				x_4,		… }
x_3	$\to$	{	x_1,		x_3,	x_4,		… }
x_4	$\to$	{ x_0,					x_5,	… }
x_5	$\to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…								
		{	x_1,	x_2,		x_4,		… }

Well, that can't be the case, since the set paired with x_1 doesn't contain x_1, but the diagonal set does contain x_1.



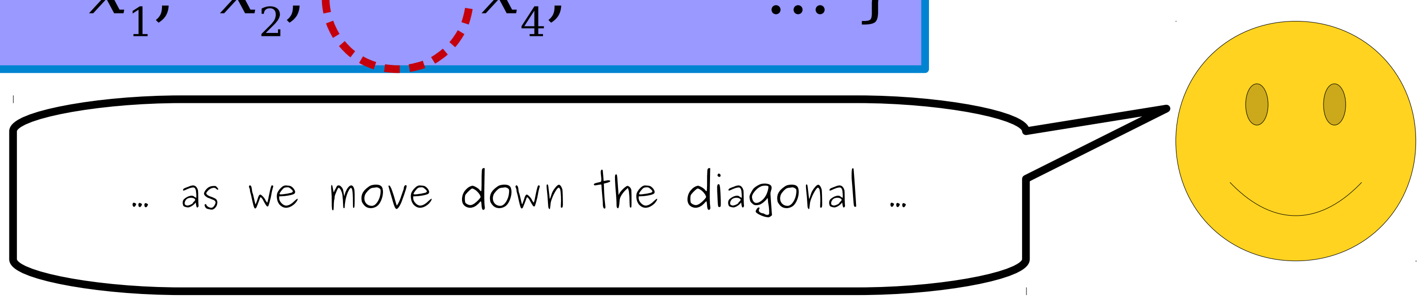
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\to$	{ x_0,		x_2,		x_4,		…	}
x_1	$\to$	{ x_0,			x_3,	x_4,		…	}
x_2	$\to$	{				x_4,		…	}
x_3	$\to$	{	x_1,		x_3,	x_4,		…	}
x_4	$\to$	{ x_0,					x_5,	…	}
x_5	$\to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…		{	x_1,	x_2,		x_4,		…	}



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

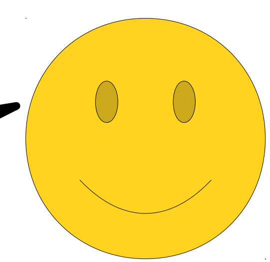
		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\to$	{ x_0,		x_2,		x_4,		…	}
x_1	$\to$	{ x_0,			x_3,	x_4,		…	}
x_2	$\to$	{				x_4,		…	}
x_3	$\to$	{	x_1,		x_3,	x_4,		…	}
x_4	$\to$	{ x_0,					x_5,	…	}
x_5	$\to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…									
		{	x_1,	x_2,		x_4,		…	}



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	{ x_0,		x_2,		x_4,		… }
$x_1 \to$	{ x_0,			x_3,	x_4,		… }
$x_2 \to$	{				x_4,		… }
$x_3 \to$	{	x_1,		x_3,	x_4,		… }
$x_4 \to$	{ x_0,					x_5,	… }
$x_5 \to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…	{	x_1,	x_2,		x_4,		… }

… we always see that there's a mismatch between the set in that row …



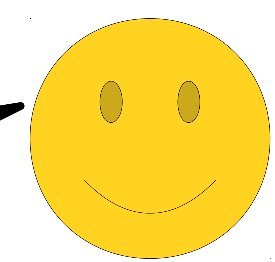
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\to$ {	x_0,		x_2,		x_4,		…	}
x_1	$\to$ {	x_0,			x_3,	x_4,		…	}
x_2	$\to$ {					x_4,		…	}
x_3	$\to$ {		x_1,		x_3,	x_4,		…	}
x_4	$\to$ {	x_0,					x_5,	…	}
x_5	$\to$ {	x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…	{		x_1,	x_2,		x_4,		…	}

… and the diagonal set we built.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
x_0	$\to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
x_1	$\to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
x_2	$\to$	$\{$				$x_4,$		$\dots \}$
x_3	$\to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
x_4	$\to$	$\{ x_0,$					$x_5,$	$\dots \}$
x_5	$\to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$								
		$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$



As a result, we know that this new set isn't paired with anything.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	{ x_0,		x_2,		x_4,		… }
$x_1 \to$	{ x_0,			x_3,	x_4,		… }
$x_2 \to$	{				x_4,		… }
$x_3 \to$	{	x_1,		x_3,	x_4,		… }
$x_4 \to$	{ x_0,					x_5,	… }
$x_5 \to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…							
Set D $\to$	{	x_1,	x_2,		x_4,		… }

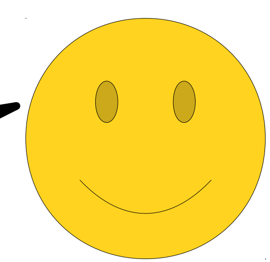
Formally speaking, if we call our set D…

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
$x_0 \longrightarrow$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \longrightarrow$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \longrightarrow$	$\{$				$x_4,$		$\dots \}$
$x_3 \longrightarrow$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \longrightarrow$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \longrightarrow$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$							
Set $D \longrightarrow$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

$f : S \to \wp(S)$

... and this particular function is f...



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	$\dots$

$x_0 \longrightarrow \{\, x_0, \; x_2, \; x_4, \; \dots \,\}$

$x_1 \longrightarrow \{\, x_0, \; x_3, \; x_4, \; \dots \,\}$

$x_2 \longrightarrow \{\, x_4, \; \dots \,\}$

$x_3 \longrightarrow \{\, x_1, \; x_3, \; x_4, \; \dots \,\}$

$x_4 \longrightarrow \{\, x_0, \; x_5, \; \dots \,\}$

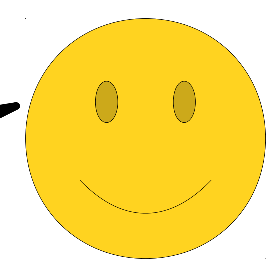
$x_5 \longrightarrow \{\, x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \,\}$

$\dots$

$f : S \to \wp(S)$

Set D $\longrightarrow$ $\boxed{\{\, x_1, \; x_2, \; x_4, \; \dots \,\}}$

Then we see that $f(x) \neq D$ for any $x \in S$, meaning that f isn't surjective!



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$$x_0 \longrightarrow \{\, x_0,\quad x_2,\quad x_4,\quad \dots \,\}$$

$$x_1 \longrightarrow \{\, x_0,\quad x_3,\ x_4,\quad \dots \,\}$$

$$x_2 \longrightarrow \{\, x_4,\quad \dots \,\}$$

$$x_3 \longrightarrow \{\, x_1,\quad x_3,\ x_4,\quad \dots \,\}$$

$$x_4 \longrightarrow \{\, x_0,\quad x_5,\ \dots \,\}$$

$$x_5 \longrightarrow \{\, x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \dots \,\}$$

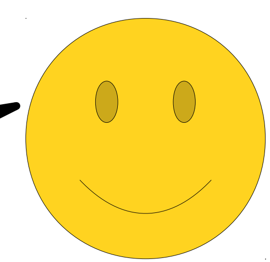
$$\dots$$

$$f : S \to \wp(S)$$

$$\boxed{\{\, x_1,\ x_2,\quad x_4,\quad \dots \,\}}$$

Set D

The beauty behind this construction is that it doesn't matter what choice of f we make.

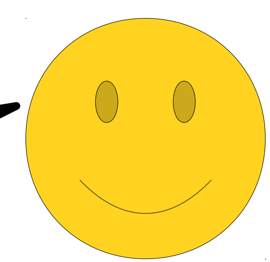


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longrightarrow$ {	x_0,		x_2,		x_4,		… }
$x_1 \longrightarrow$ {	x_0,			x_3,	x_4,		… }
$x_2 \longrightarrow$ {					x_4,		… }
$x_3 \longrightarrow$ {		x_1,		x_3,	x_4,		… }
$x_4 \longrightarrow$ {	x_0,					x_5,	… }
$x_5 \longrightarrow$ {	x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…							
Set D {		x_1,	x_2,		x_4,		… }

$f : S \to \wp(S)$

If we take the diagonal and flip it, we'll always find a set that nothing maps to.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\ x_2,\ x_4,\ \dots\ \}$

$x_1 \longrightarrow \{\ x_0,\ x_3,\ x_4,\ \dots\ \}$

$x_2 \longrightarrow \{\ x_4,\ \dots\ \}$

$x_3 \longrightarrow \{\ x_1,\ x_3,\ x_4,\ \dots\ \}$

$x_4 \longrightarrow \{\ x_0,\ x_5,\ \dots\ \}$

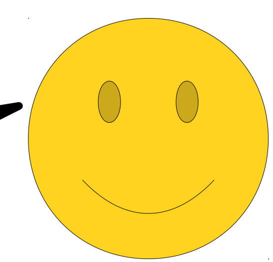
$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \dots\ \}$

…

$f : S \to \wp(S)$

Set D $\longrightarrow$ $\boxed{\{\ x_1,\ x_2,\ x_4,\ \dots\ \}}$

Our challenge now is to convert this picture into a rigorous proof.

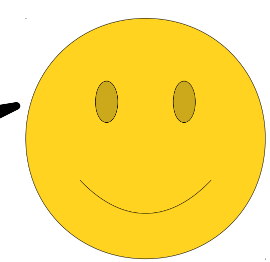


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	$\dots$	
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots$	$\}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots$	$\}$
$x_2 \to$	$\{$				$x_4,$		$\dots$	$\}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots$	$\}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots$	$\}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots$	$\}$
$\dots$								
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots$	$\}$

$f : S \to \wp(S)$

To do that, we're going to need to find a way to rigorously define the set D.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
…							
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

$f : S \to \wp(S)$

Once we've done that, we can write a formal proof that $f(x) \neq D$ for any $x \in S$.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$$x_0 \longrightarrow \{\, x_0,\ x_2,\ x_4,\ \dots \,\}$$

$$x_1 \longrightarrow \{\, x_0,\ x_3,\ x_4,\ \dots \,\}$$

$$x_2 \longrightarrow \{\, x_4,\ \dots \,\}$$

$$x_3 \longrightarrow \{\, x_1,\ x_3,\ x_4,\ \dots \,\}$$

$$x_4 \longrightarrow \{\, x_0,\ x_5,\ \dots \,\}$$

$$x_5 \longrightarrow \{\, x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \dots \,\}$$

$$\dots$$

$$\boxed{\{\, x_1,\ x_2,\ x_4,\ \dots \,\}}$$

Set D

So... what is this set D?

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	$\dots$

$$x_0 \longrightarrow \{\, x_0, \quad x_2, \quad x_4, \quad \dots \,\}$$

$$x_1 \longrightarrow \{\, x_0, \quad x_3, \; x_4, \quad \dots \,\}$$

$$x_2 \longrightarrow \{\, x_4, \quad \dots \,\}$$

$$x_3 \longrightarrow \{\, x_1, \quad x_3, \; x_4, \quad \dots \,\}$$

$$x_4 \longrightarrow \{\, x_0, \quad x_5, \; \dots \,\}$$

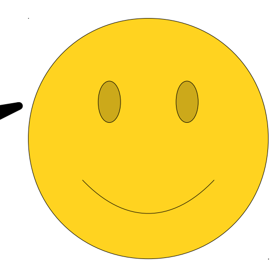
$$x_5 \longrightarrow \{\, x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \,\}$$

$$\dots$$

$$\boxed{\{\, x_1, \; x_2, \quad x_4, \quad \dots \,\}}$$

Set D

Since sets are uniquely defined by the elements they contain, a reasonable first step is to see what's in D.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
…							
Set D $\to$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

Although in this discussion I'm going to use the example f given here, this will work for any choice of f.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
…							
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

Let's look at x_1, for example.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\, x_0, \quad x_2, \quad x_4, \quad \ldots \,\}$

$x_1 \longrightarrow \{\, x_0, \quad x_3, \; x_4, \quad \ldots \,\}$

$x_2 \longrightarrow \{\, x_4, \quad \ldots \,\}$

$x_3 \longrightarrow \{\, x_1, \quad x_3, \; x_4, \quad \ldots \,\}$

$x_4 \longrightarrow \{\, x_0, \quad x_5, \; \ldots \,\}$

$x_5 \longrightarrow \{\, x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \ldots \,\}$

$\ldots$

$\{\, x_1, \; x_2, \quad x_4, \quad \ldots \,\}$

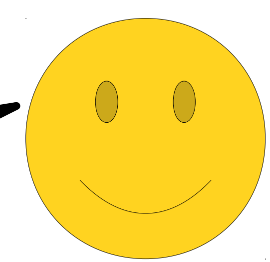
Set D

We included x_1 in this set because x_1 isn't on the main diagonal of this grid.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

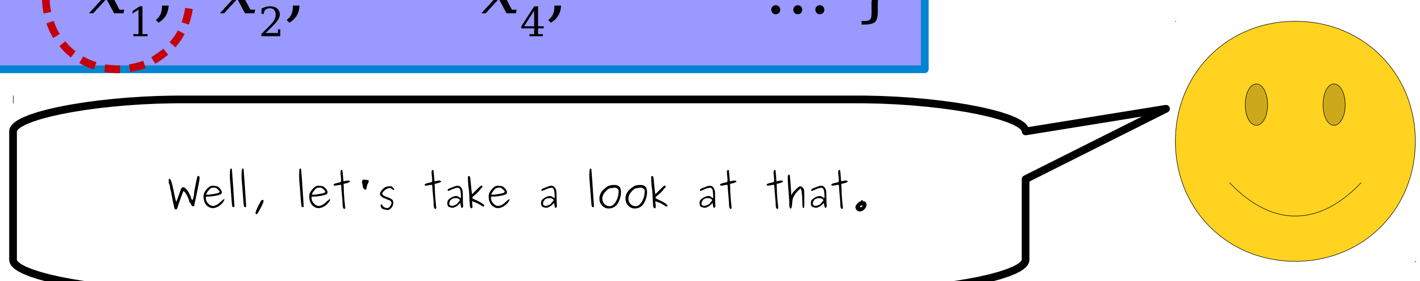
	x_0	x_1	x_2	x_3	x_4	x_5	…	
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		…	$\}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		…	$\}$
$x_2 \to$	$\{$				$x_4,$		…	$\}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		…	$\}$
$x_4 \to$	$\{ x_0,$					$x_5,$	…	$\}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	…	$\}$
…								
Set $D \to$	$\{$	$x_1,$	$x_2,$		$x_4,$		…	$\}$

How can we describe this more rigorously, without appealing to this picture?



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

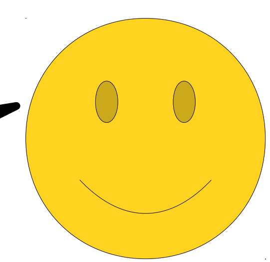
		x_0	x_1	x_2	x_3	x_4	x_5	$\dots$	
x_0	$\to$	$\{\ x_0,$		$x_2,$		$x_4,$		$\dots$	$\}$
x_1	$\to$	$\{\ x_0,$			$x_3,$	$x_4,$		$\dots$	$\}$
x_2	$\to$	$\{$				$x_4,$		$\dots$	$\}$
x_3	$\to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots$	$\}$
x_4	$\to$	$\{\ x_0,$					$x_5,$	$\dots$	$\}$
x_5	$\to$	$\{\ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots$	$\}$
$\dots$									
Set D	$\to$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots$	$\}$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	{ x_0,		x_2,		x_4,		… }
$x_1 \to$	{ x_0,			x_3,	x_4,		… }
$x_2 \to$	{				x_4,		… }
$x_3 \to$	{	x_1,		x_3,	x_4,		… }
$x_4 \to$	{ x_0,					x_5,	… }
$x_5 \to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…							
Set D $\to$	{	x_1,	x_2,		x_4,		… }

First, notice that we've highlighted this particular set. What set is this?

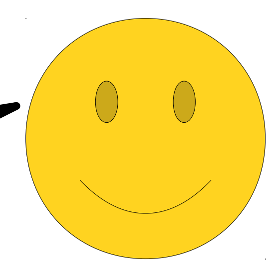


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

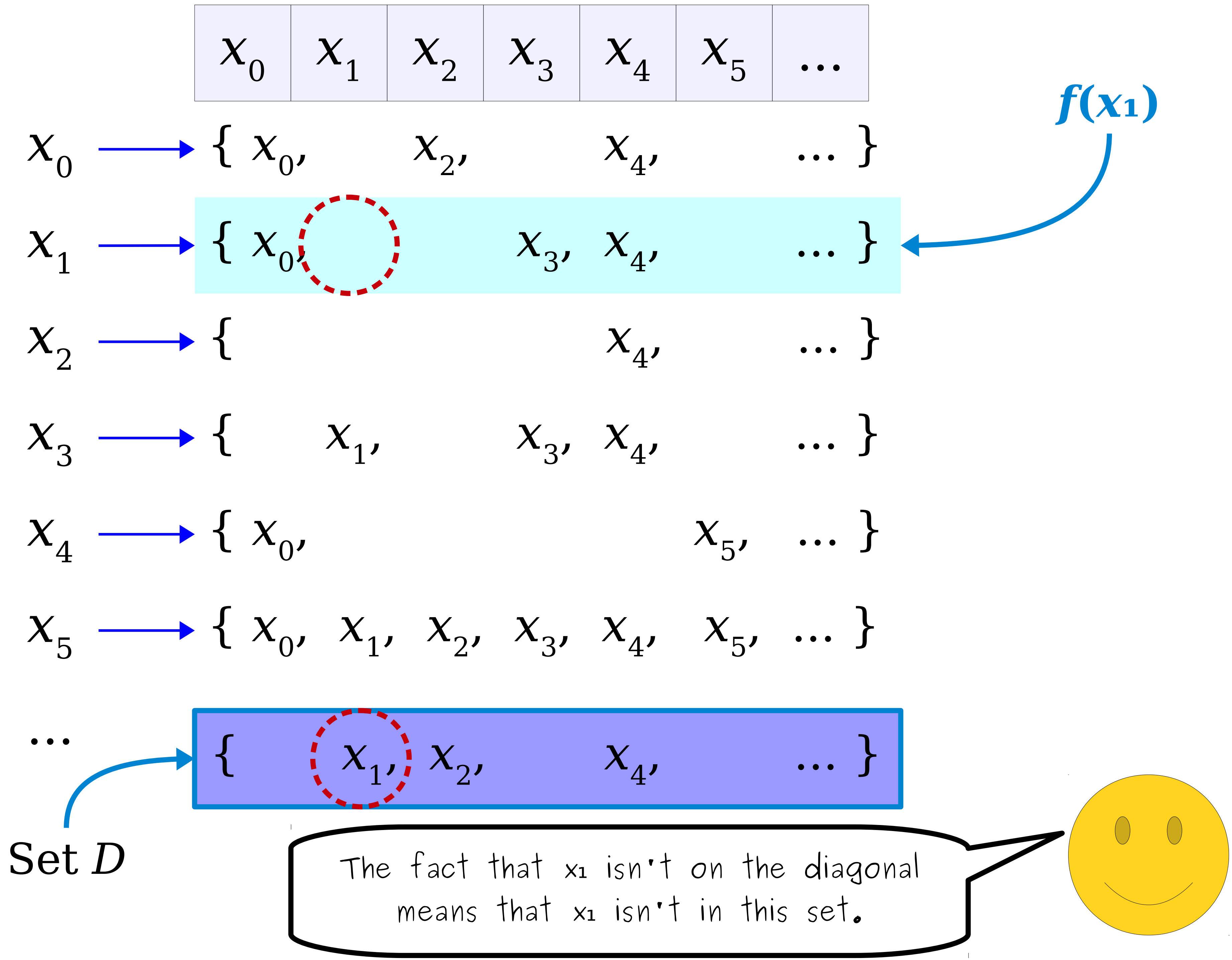
	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$							
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

$f(x_1)$

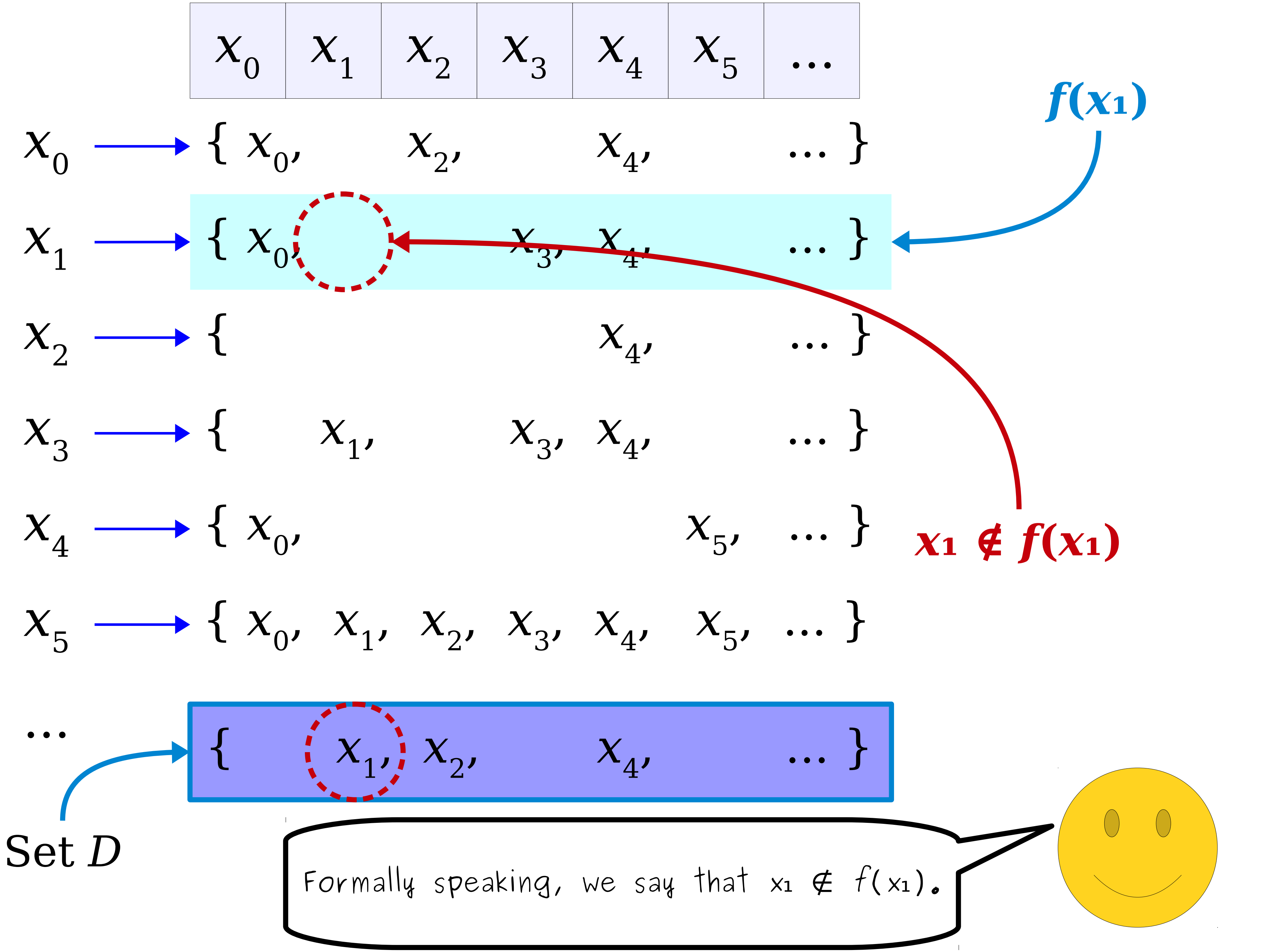
Mathematically, this highlighted row is $f(x_1)$, the set paired with x_1.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$f(x_1)$

$x_0 \to \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \to \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \to \{ x_4, \quad \dots \}$

$x_3 \to \{ x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \to \{ x_0, \quad x_5, \quad \dots \}$

$x_1 \notin f(x_1)$

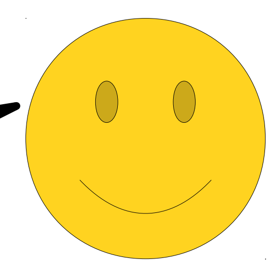
$x_5 \to \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$

$\{ x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

So, because $x_1 \notin f(x_1)$, we included x_1 in D.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

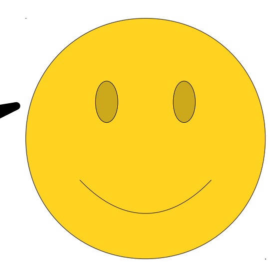
	x_0	x_1	x_2	x_3	x_4	x_5	...
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\dots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$							
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

Similarly, let's take a look at why x_2 is in our set.

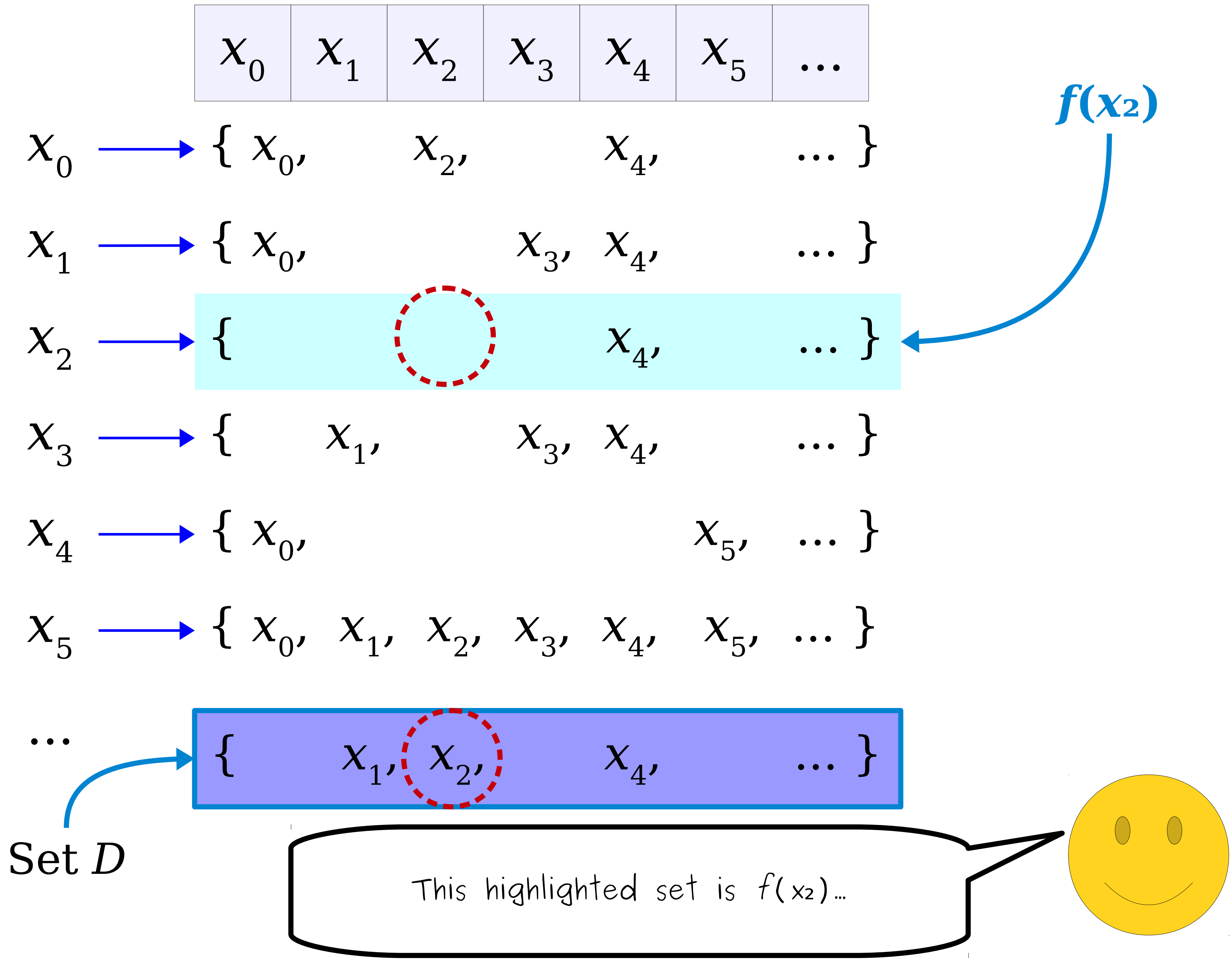
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	...	
x_0	$\to$ {	x_0,		x_2,		x_4,		...	}
x_1	$\to$ {	x_0,			x_3,	x_4,		...	}
x_2	$\to$ {					x_4,		...	}
x_3	$\to$ {		x_1,		x_3,	x_4,		...	}
x_4	$\to$ {	x_0,					x_5,	...	}
x_5	$\to$ {	x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	...	}
...									
Set D	$\to$ {		x_1,	x_2,		x_4,		...	}

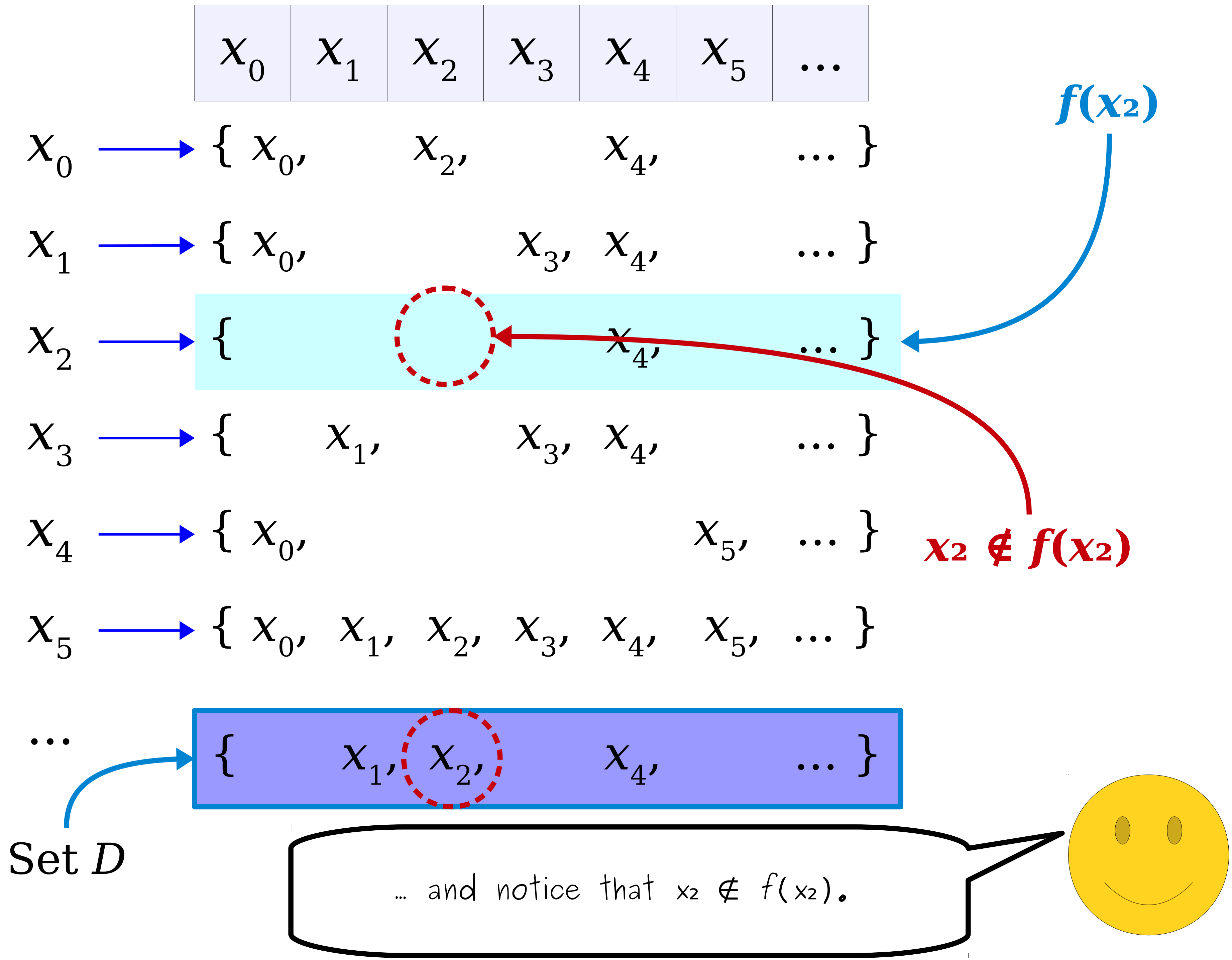
We included it because it's missing off the diagonal.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



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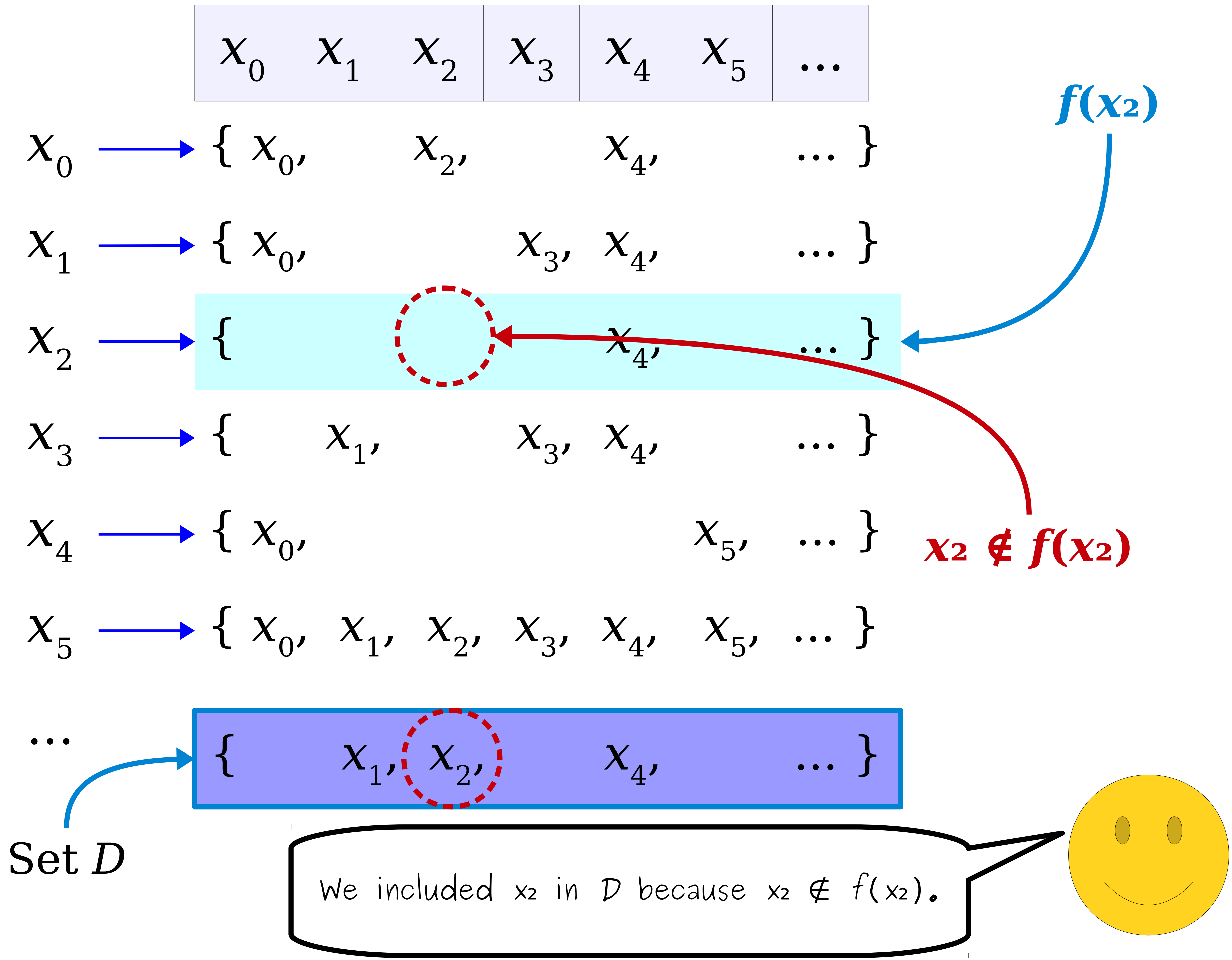

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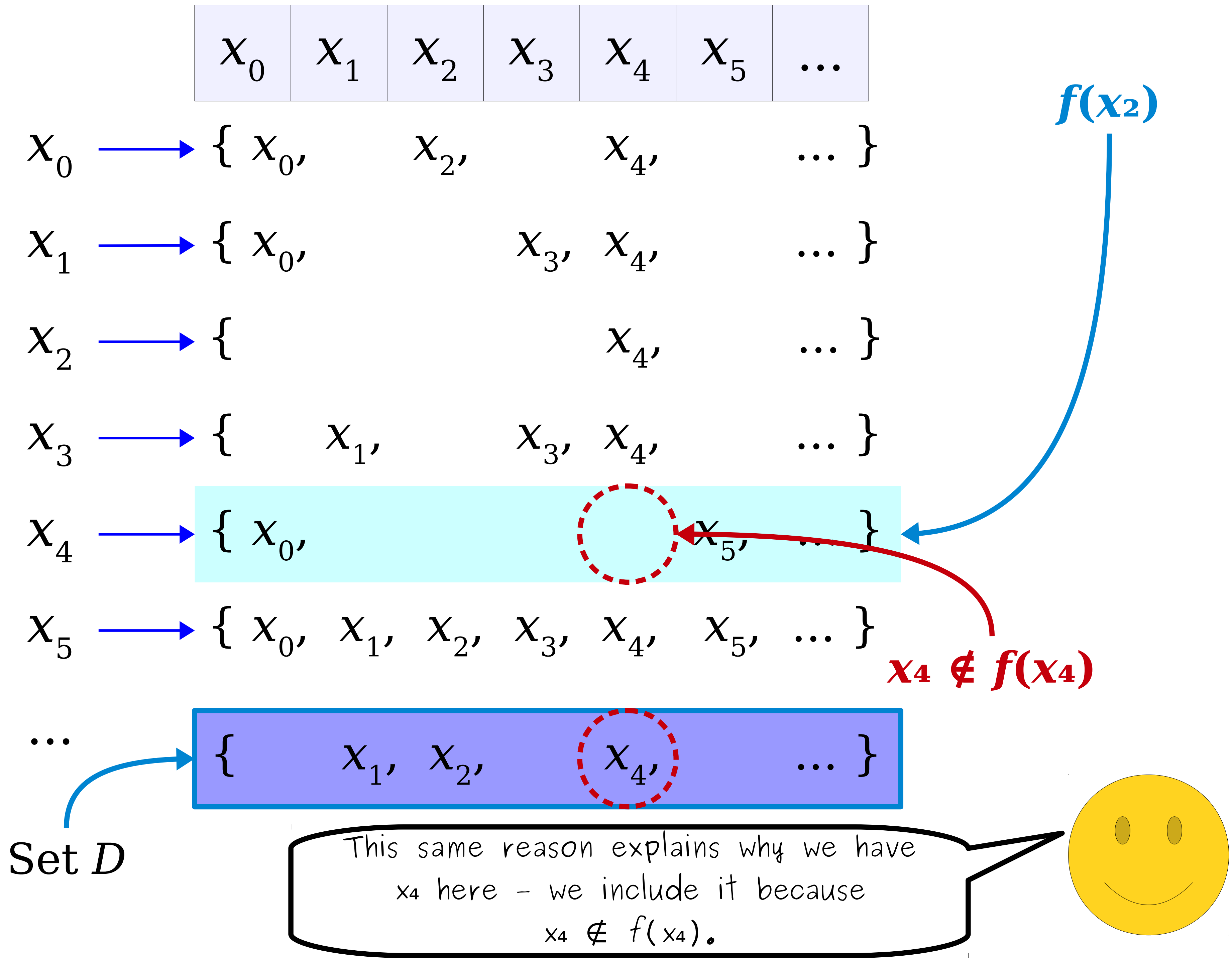
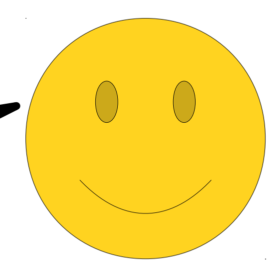


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	$\dots$	
x_0	$\to$ {	x_0,		x_2,		x_4,		$\dots$	}
x_1	$\to$ {	x_0,			x_3,	x_4,		$\dots$	}
x_2	$\to$ {					x_4,		$\dots$	}
x_3	$\to$ {		x_1,		x_3,	x_4,		$\dots$	}
x_4	$\to$ {	x_0,					x_5,	$\dots$	}
x_5	$\to$ {	x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	$\dots$	}
$\dots$									
Set D	$\to$ {		x_1,	x_2,		x_4,		$\dots$	}

So it seems like we have a general pattern going here.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

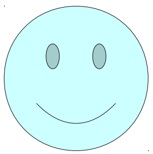
	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longrightarrow$	{ x_0,		x_2,		x_4,		… }
$x_1 \longrightarrow$	{ x_0,			x_3,	x_4,		… }
$x_2 \longrightarrow$	{				x_4,		… }
$x_3 \longrightarrow$	{	x_1,		x_3,	x_4,		… }
$x_4 \longrightarrow$	{ x_0,					x_5,	… }
$x_5 \longrightarrow$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…							
Set D $\longrightarrow$	{	x_1,	x_2,		x_4,		… }

Specifically, if we find an x where $x \notin f(x)$, then it ends up in D.

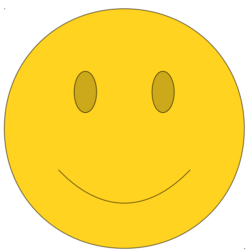
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…	
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		…	$\}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		…	$\}$
$x_2 \to$	$\{$				$x_4,$		…	$\}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		…	$\}$
$x_4 \to$	$\{ x_0,$							$\}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$					
…								
Set D $\to$	$\{$	$x_1,$	$x_2,$		$x_4,$		…	$\}$

(That's actually really important, so make a note of this for later on.)



Specifically, if we find an x where $x \notin f(x)$, then it ends up in D.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$

$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$

$x_2 \longrightarrow \{ x_4, \dots \}$

$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$

$x_4 \longrightarrow \{ x_0, x_5, \dots \}$

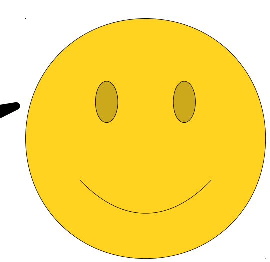
$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

$\boxed{\{ x_1, x_2, x_4, \dots \}}$

Set D

So what about the elements that aren't in our set D?



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	$\ldots$

$$x_0 \longrightarrow \{\, x_0, \quad x_2, \quad x_4, \quad \ldots \,\}$$

$$x_1 \longrightarrow \{\, x_0, \quad x_3, \quad x_4, \quad \ldots \,\}$$

$$x_2 \longrightarrow \{\, x_4, \quad \ldots \,\}$$

$$x_3 \longrightarrow \{\, x_1, \quad x_3, \quad x_4, \quad \ldots \,\}$$

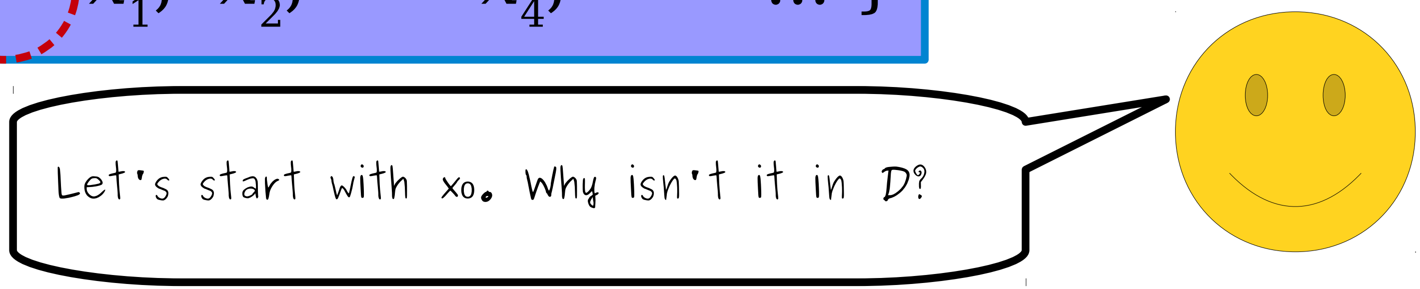
$$x_4 \longrightarrow \{\, x_0, \quad x_5, \quad \ldots \,\}$$

$$x_5 \longrightarrow \{\, x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \ldots \,\}$$

$$\ldots$$

$$\{\, x_1, \quad x_2, \quad x_4, \quad \ldots \,\}$$

Set D

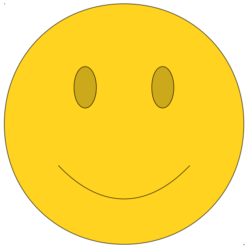


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	{ x_0,		x_2,		x_4,		… }
$x_1 \to$	{ x_0,			x_3,	x_4,		… }
$x_2 \to$	{				x_4,		… }
$x_3 \to$	{	x_1,		x_3,	x_4,		… }
$x_4 \to$	{ x_0,					x_5,	… }
$x_5 \to$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	… }
…	{	x_1,	x_2,		x_4,		… }

Set D

Well, it's not in D because x_0 does appear on the diagonal.

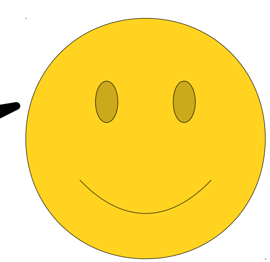


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\ldots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\ldots \}$
$x_2 \to$	$\{$				$x_4,$		$\ldots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\ldots \}$
$x_4 \to$	$\{ x_0,$					$x_5,$	$\ldots \}$
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\ldots \}$
…							
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\ldots \}$

$f(x_0)$

This particular row is $f(x_0)$…



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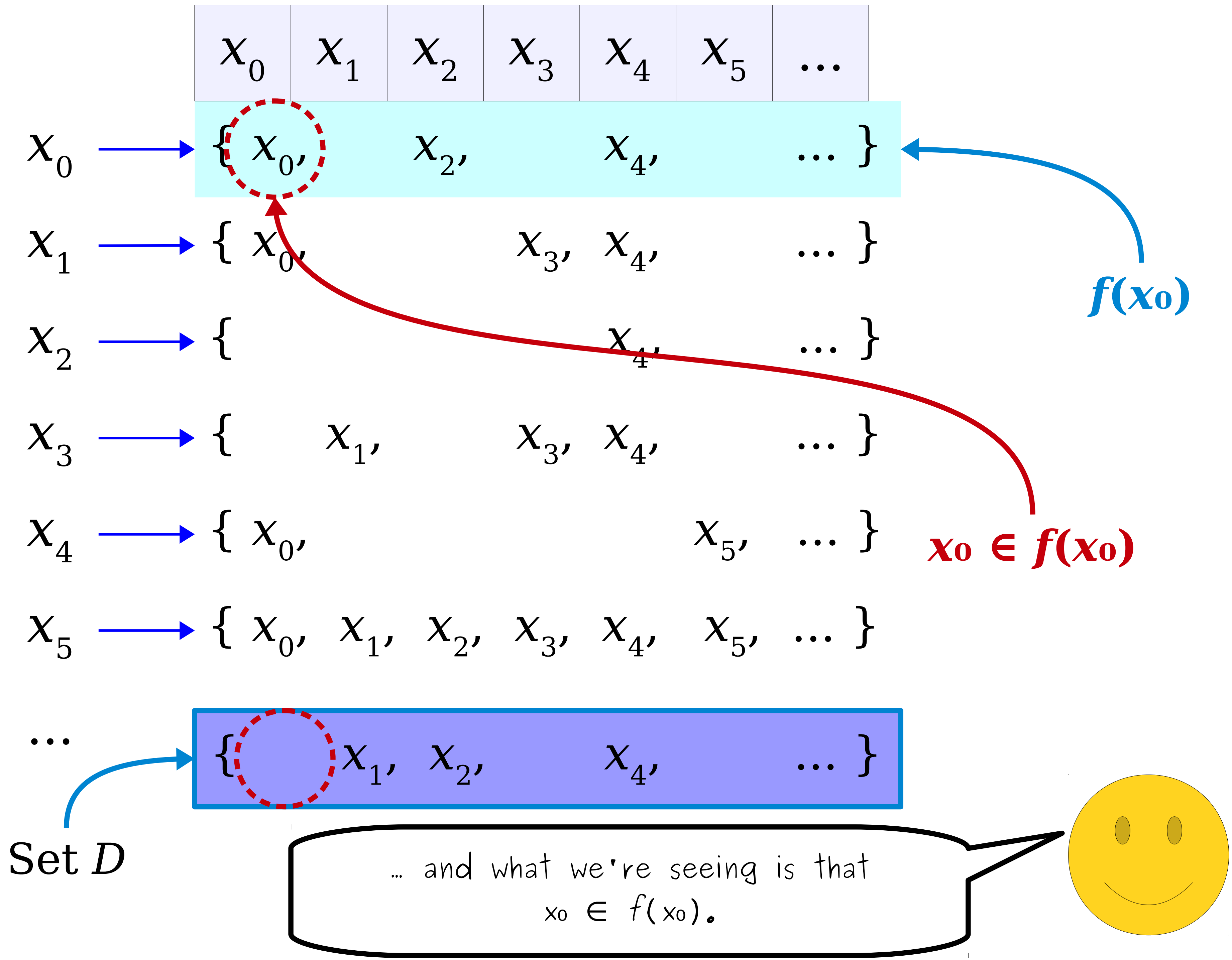


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \; x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \; x_4, \quad \dots \}$ ← $f(x_3)$

$x_4 \longrightarrow \{ x_0, \quad x_5, \; \dots \}$

$x_5 \longrightarrow \{ x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \}$

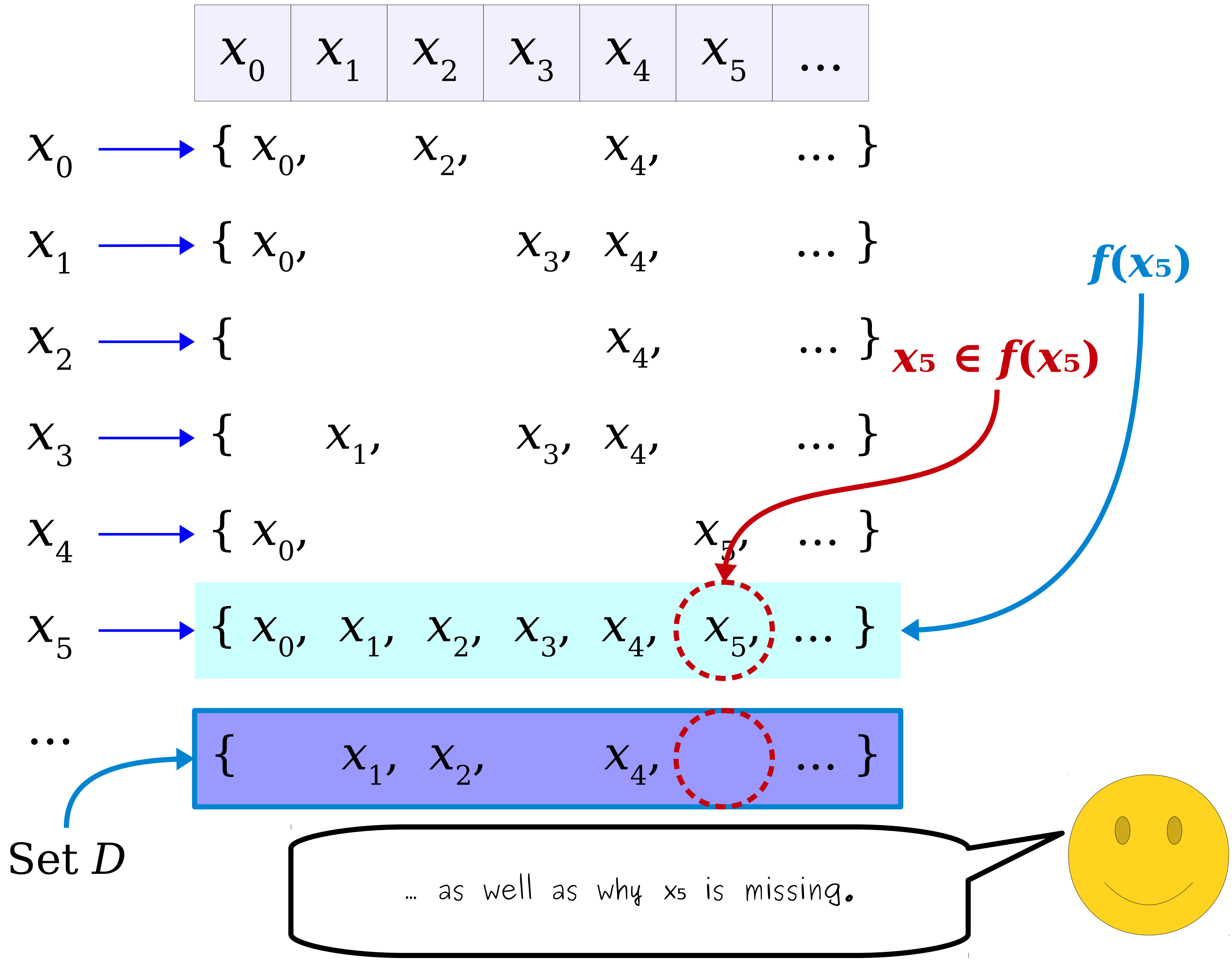
$x_3 \in f(x_3)$

…

Set D $\longrightarrow \{ \quad x_1, \; x_2, \quad x_4, \quad \dots \}$

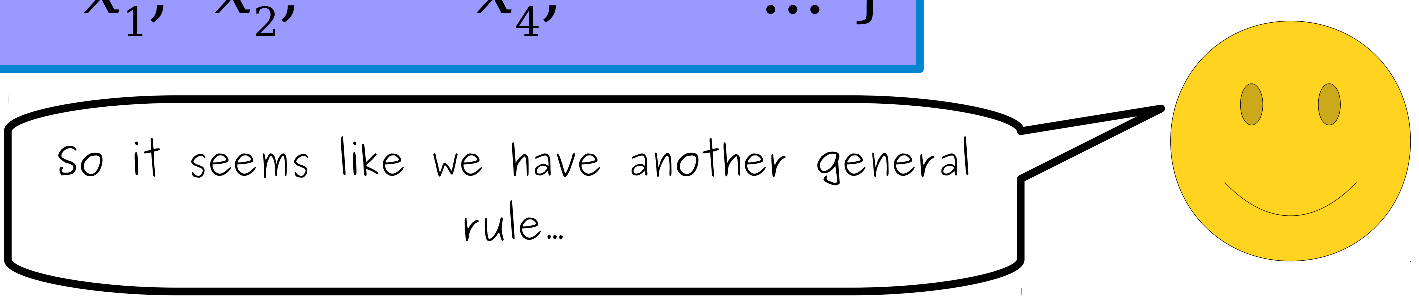
Similar reasoning explains why we didn't include x₃ in our set…

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…	
$x_0 \longrightarrow$	{ x_0,		x_2,		x_4,		…	}
$x_1 \longrightarrow$	{ x_0,			x_3,	x_4,		…	}
$x_2 \longrightarrow$	{				x_4,		…	}
$x_3 \longrightarrow$	{	x_1,		x_3,	x_4,		…	}
$x_4 \longrightarrow$	{ x_0,					x_5,	…	}
$x_5 \longrightarrow$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…								
Set D $\longrightarrow$	{	x_1,	x_2,		x_4,		…	}



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

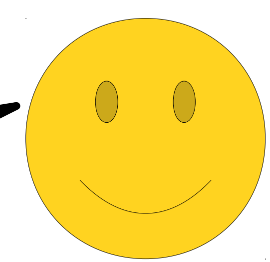
	x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
$x_0 \longrightarrow$	$\{\ x_0,$		$x_2,$		$x_4,$		$\dots\ \}$
$x_1 \longrightarrow$	$\{\ x_0,$			$x_3,$	$x_4,$		$\dots\ \}$
$x_2 \longrightarrow$	$\{$				$x_4,$		$\dots\ \}$
$x_3 \longrightarrow$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots\ \}$
$x_4 \longrightarrow$	$\{\ x_0,$					$x_5,$	$\dots\ \}$
$x_5 \longrightarrow$	$\{\ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots\ \}$
$\dots$							
Set D $\longrightarrow$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots\ \}$

…the set D excludes every x where $x \in f(x)$.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{\ x_0,$		$x_2,$		$x_4,$		$\dots\ \}$
$x_1 \to$	$\{\ x_0,$			$x_3,$	$x_4,$		$\dots\ \}$
$x_2 \to$	$\{$				$x_4,$		$\dots\ \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		$\dots\ \}$
$x_4 \to$	$\{\ x_0,$					$x_5,$	$\dots\ \}$
$x_5 \to$	$\{\ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots\ \}$
…							
Set D $\to$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots\ \}$

To recap…



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		
$x_4 \to$	$\{ x_0,$					$x_5,$	
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots \}$
$\dots$							
Set D $\to$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

If $x \notin f(x)$, then $x \in D$.

If we choose any x and see that $x \notin f(x)$, then $x \in D$.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longrightarrow$	$\{\ x_0,$		$x_2,$		$x_4,$		$\dots\ \}$
$x_1 \longrightarrow$	$\{\ x_0,$			$x_3,$	$x_4,$		$\dots\ \}$
$x_2 \longrightarrow$	$\{$				$x_4,$		$\dots\ \}$
$x_3 \longrightarrow$	$\{$	$x_1,$		$x_3,$	$x_4,$		
$x_4 \longrightarrow$	$\{\ x_0,$					$x_5,$	
$x_5 \longrightarrow$	$\{\ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$\dots\ \}$
…							
Set $D \longrightarrow$	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots\ \}$

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

Similarly, if we choose any x and find that $x \in f(x)$, then $x \notin D$.

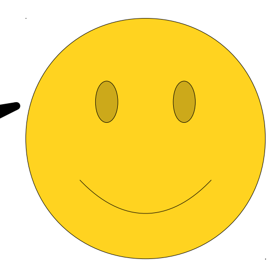
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	$\longrightarrow$	{ x_0,		x_2,		x_4,		…	}
x_1	$\longrightarrow$	{ x_0,			x_3,	x_4,		…	}
x_2	$\longrightarrow$	{				x_4,		…	}
x_3	$\longrightarrow$	{	x_1,		x_3,	x_4,			}
x_4	$\longrightarrow$	{ x_0,					x_5,		
x_5	$\longrightarrow$	{ x_0,	x_1,	x_2,	x_3,	x_4,	x_5,	…	}
…	$\longrightarrow$	{	x_1,	x_2,		x_4,		…	}

Set D

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

Collectively, this lets us define our set D purely mathematically.



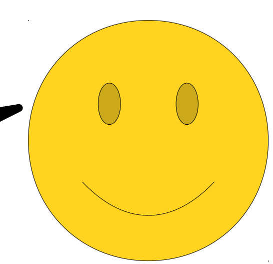
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \to$	$\{ x_0,$		$x_2,$		$x_4,$		$\dots \}$
$x_1 \to$	$\{ x_0,$			$x_3,$	$x_4,$		$\dots \}$
$x_2 \to$	$\{$				$x_4,$		$\dots \}$
$x_3 \to$	$\{$	$x_1,$		$x_3,$	$x_4,$		
$x_4 \to$	$\{ x_0,$					x_5	
$x_5 \to$	$\{ x_0,$	$x_1,$	$x_2,$	$x_3,$	$x_4,$	x_5	
…							
Set D	$\{$	$x_1,$	$x_2,$		$x_4,$		$\dots \}$

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$D = \{ x \in S \mid x \notin f(x) \}$

Specifically:
$D = \{ x \in S \mid x \notin f(x) \}$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\ \ x_2,\ \ x_4,\ \ \dots\ \}$

$x_1 \longrightarrow \{\ x_0,\ \ x_3,\ x_4,\ \ \dots\ \}$

$x_2 \longrightarrow \{\ \ x_4,\ \ \dots\ \}$

$x_3 \longrightarrow \{\ \ x_1,\ \ x_3,\ x_4,$

$x_4 \longrightarrow \{\ x_0,\ \ x_5$

$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5$

…

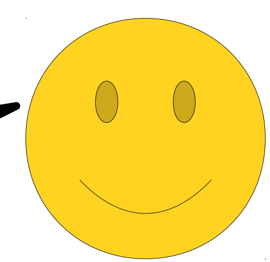
$\{\ \ x_1,\ x_2,\ \ x_4,\ \ \dots\ \}$

Set D

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$D = \{\ x \in S \mid x \notin f(x)\ \}$

Take a minute to see why this is – it's a consequence of the two above rules.



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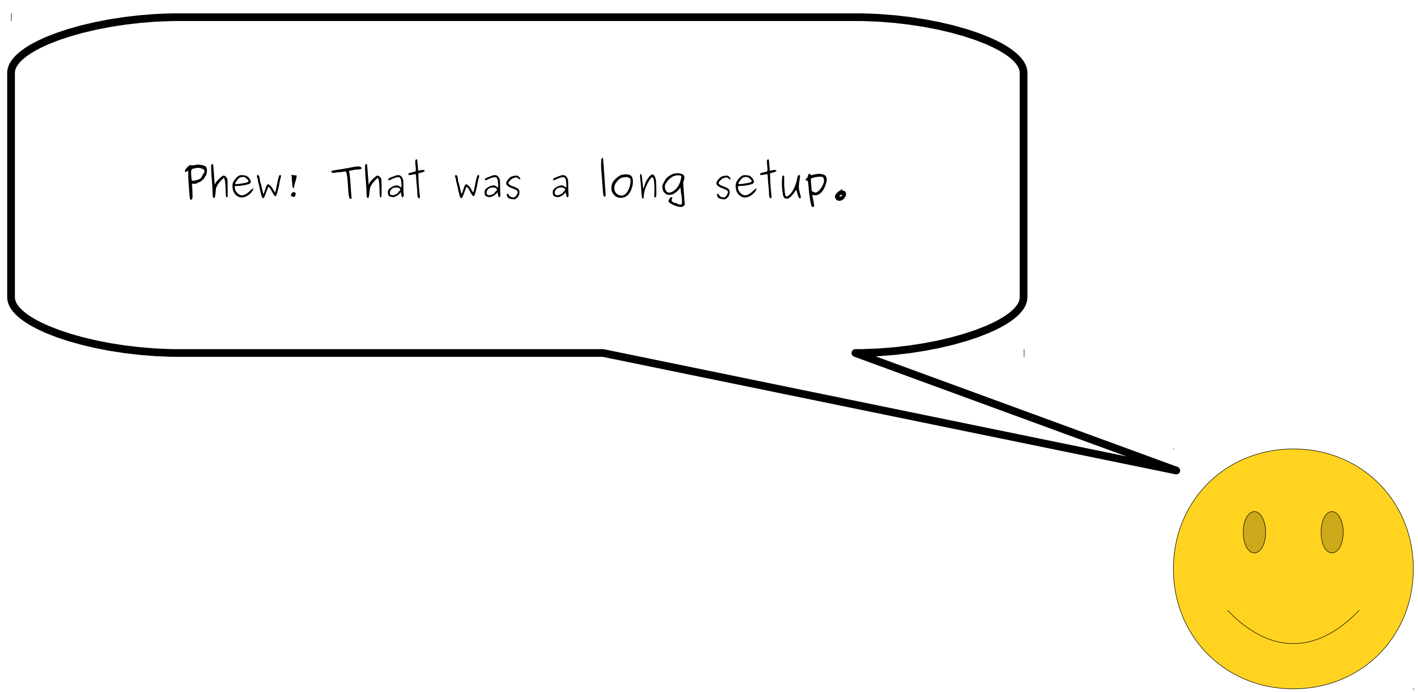
$D = \{ x \in S \mid x \notin f(x) \}$



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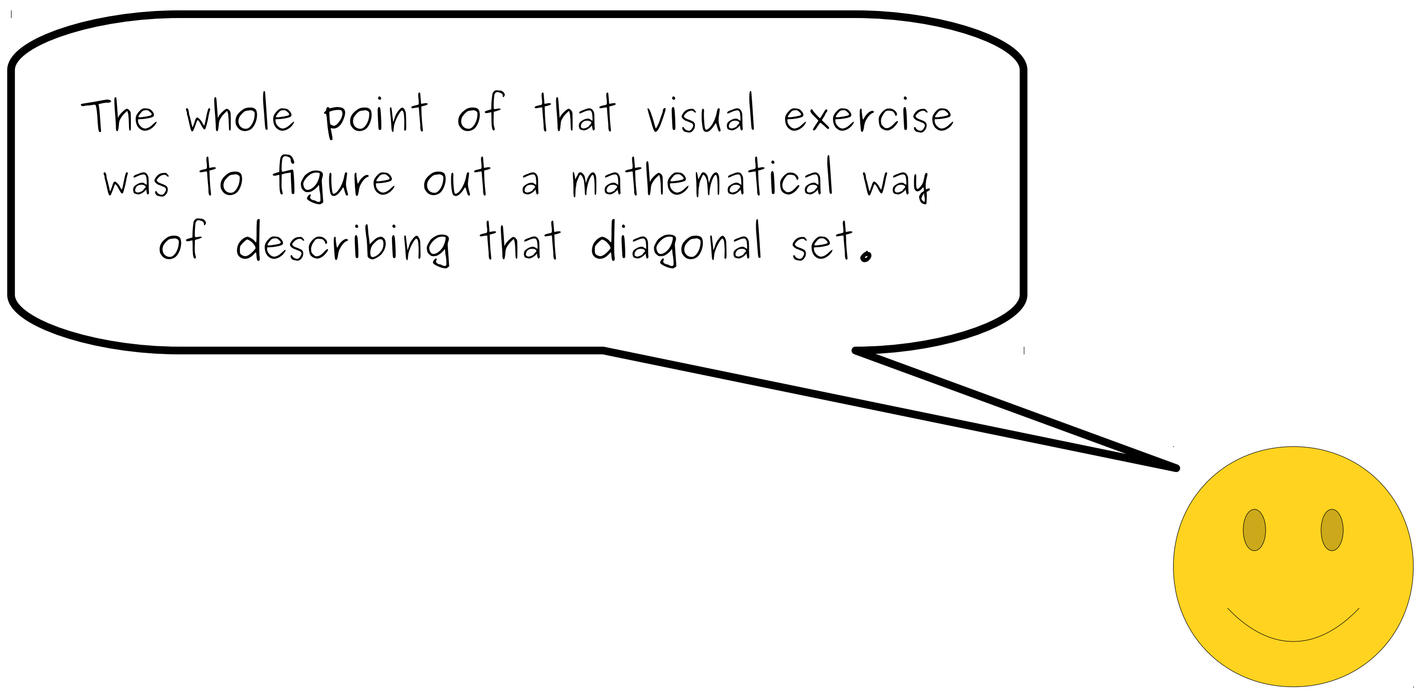

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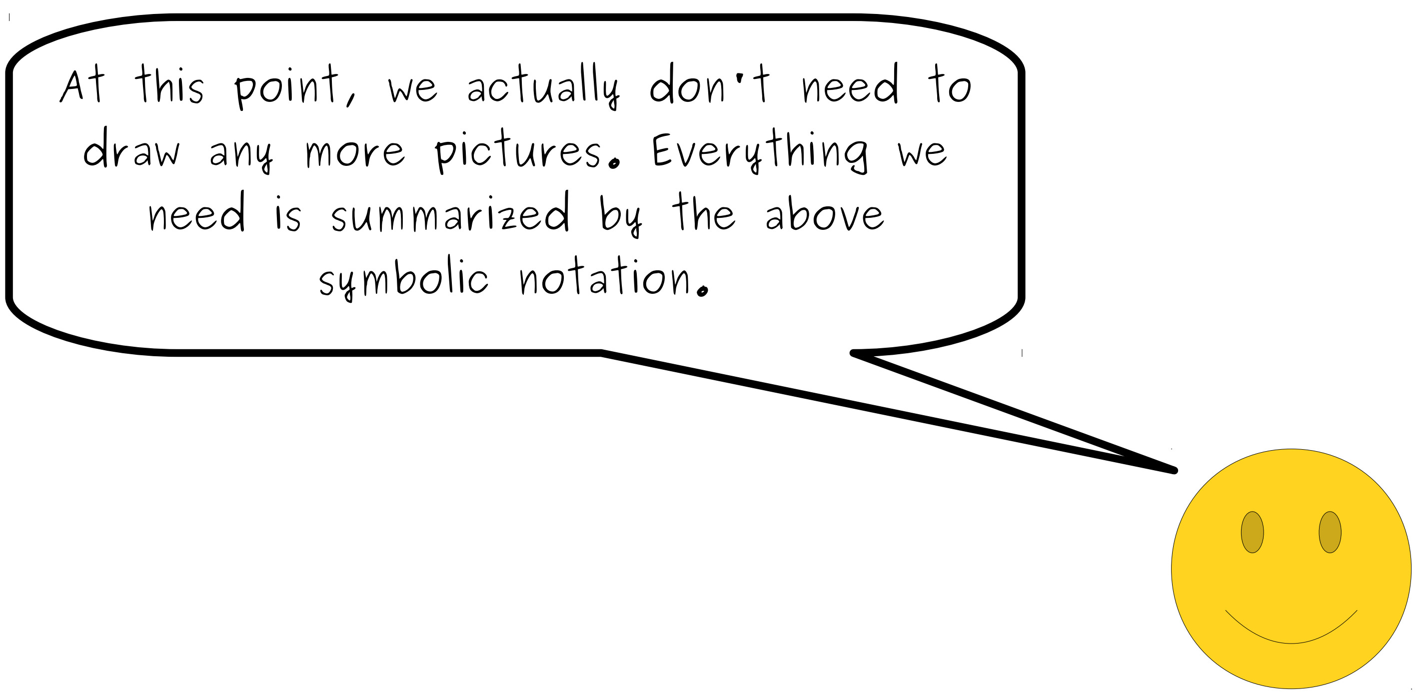
$$D = \{ x \in S \mid x \notin f(x) \}$$



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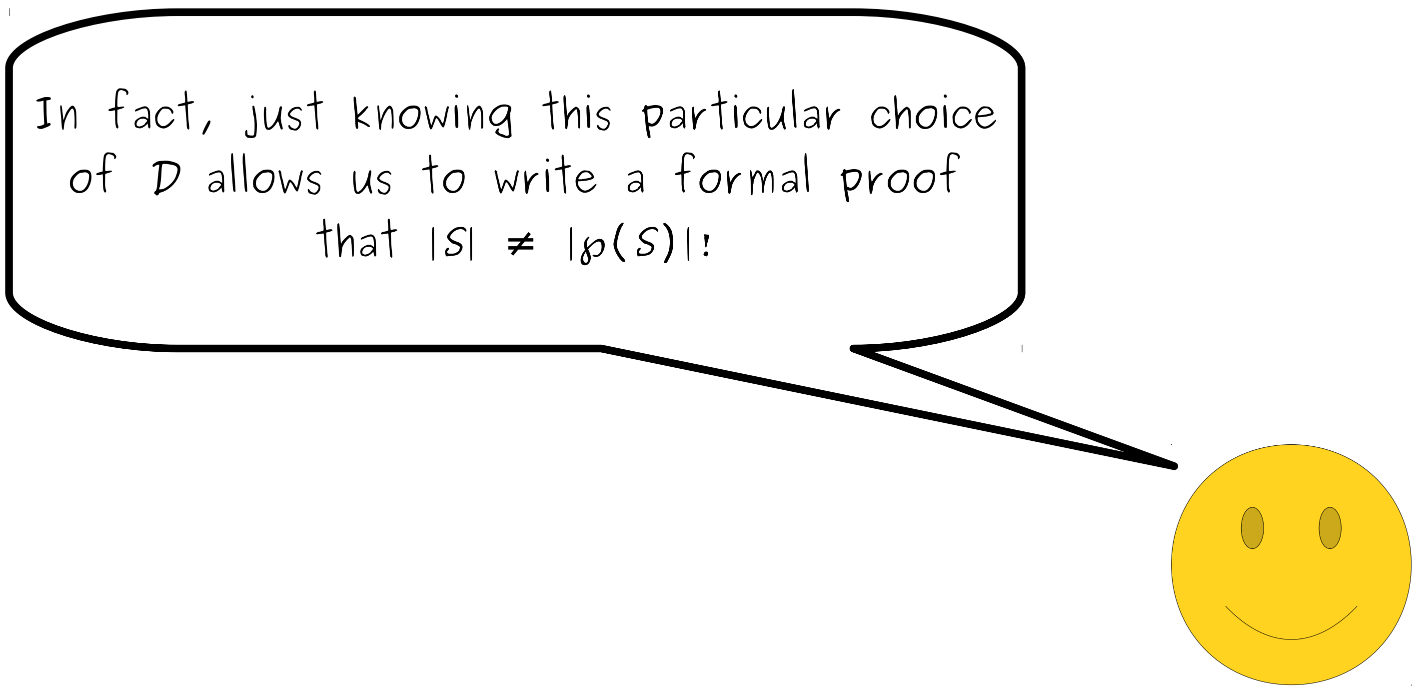
$$D = \{\, x \in S \mid x \notin f(x) \,\}$$



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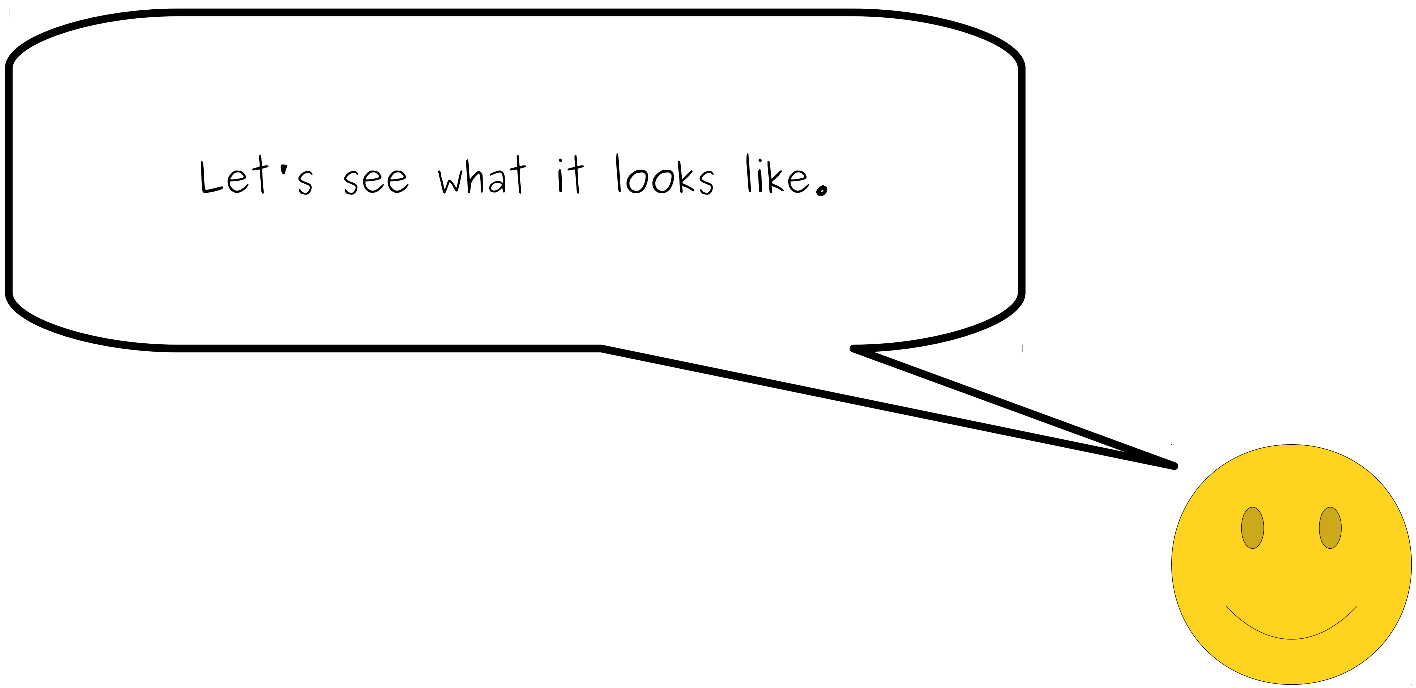
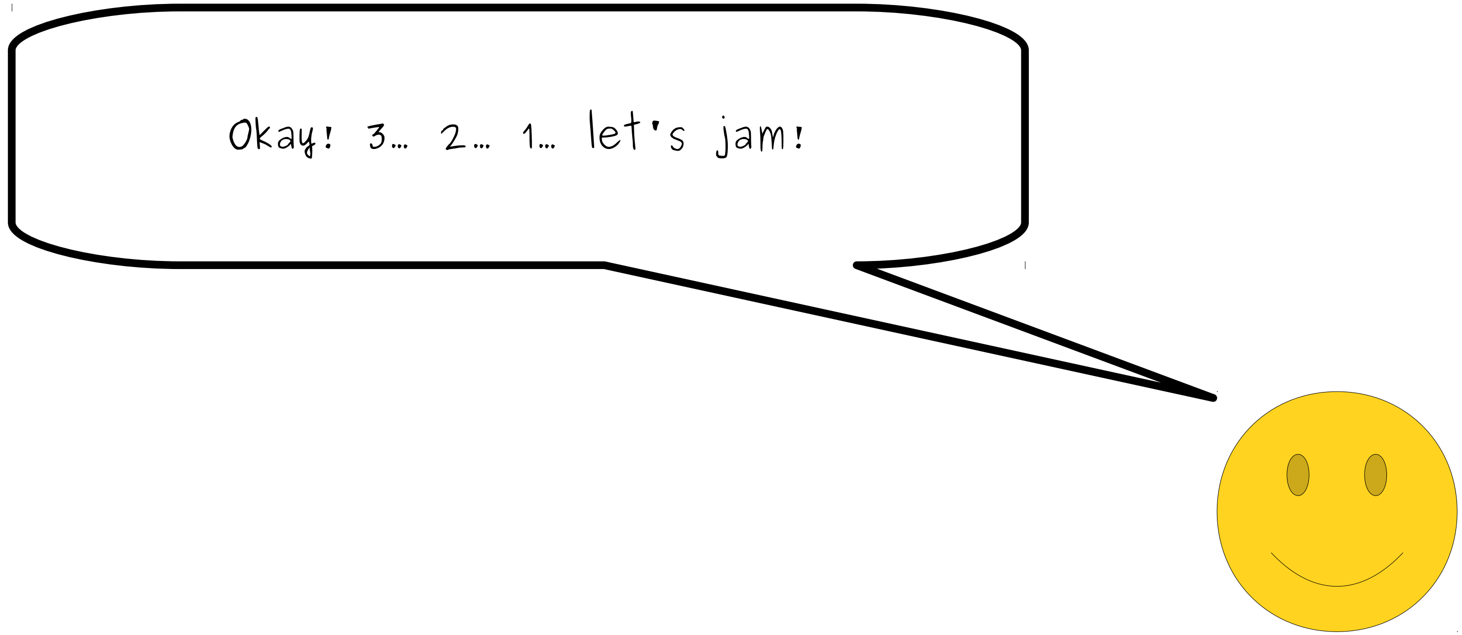
If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$$D = \{\, x \in S \mid x \notin f(x) \,\}$$



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

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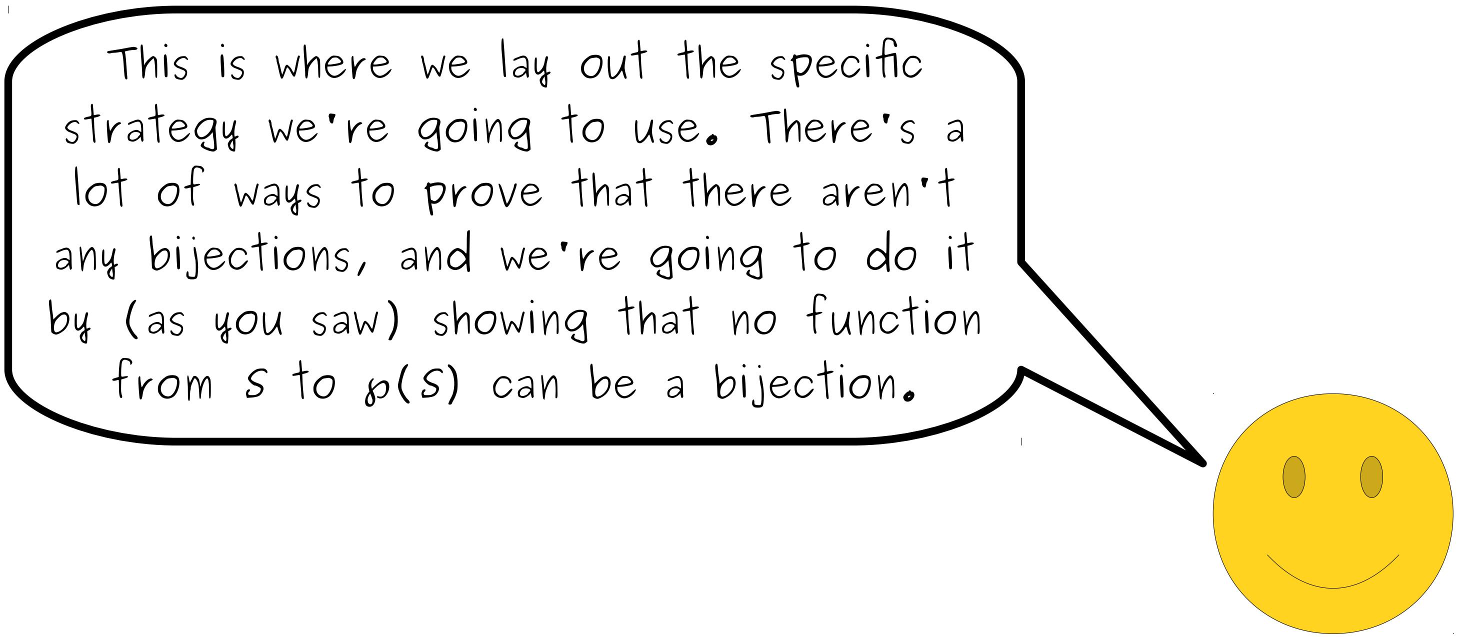

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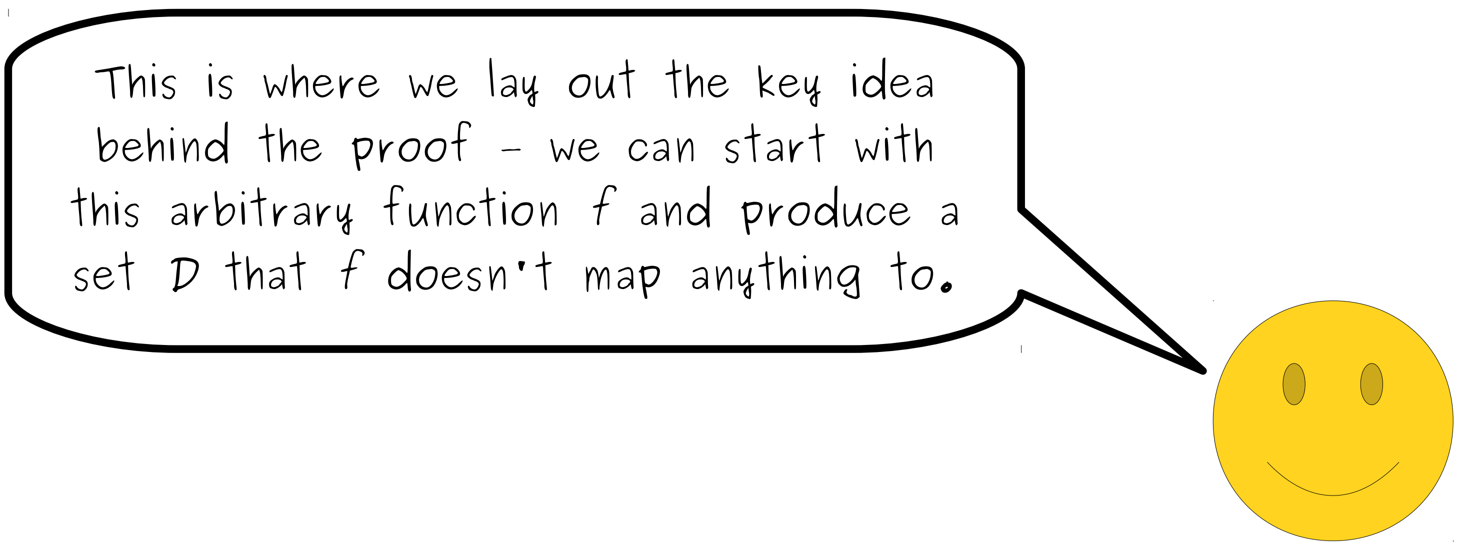

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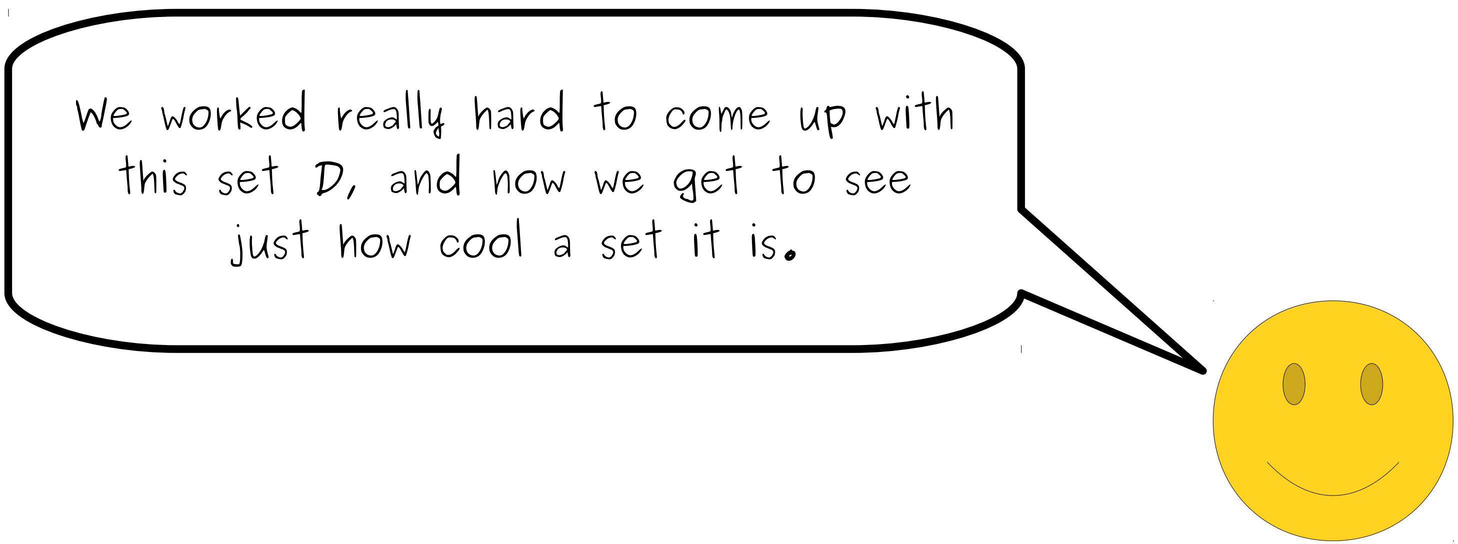

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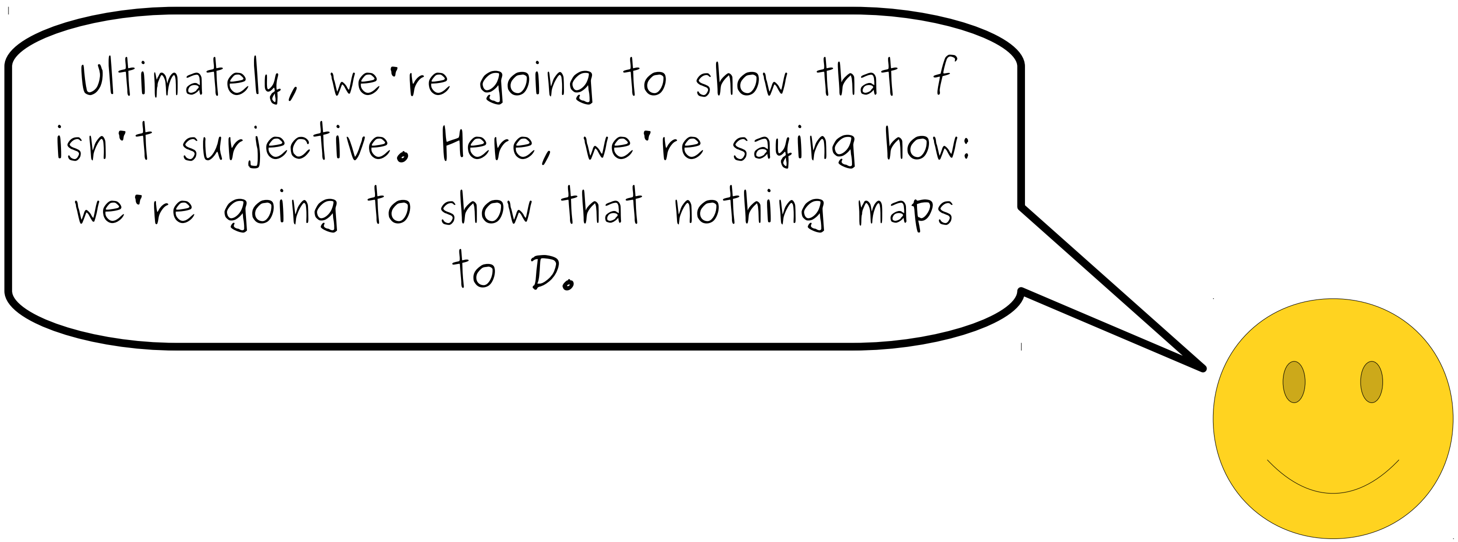

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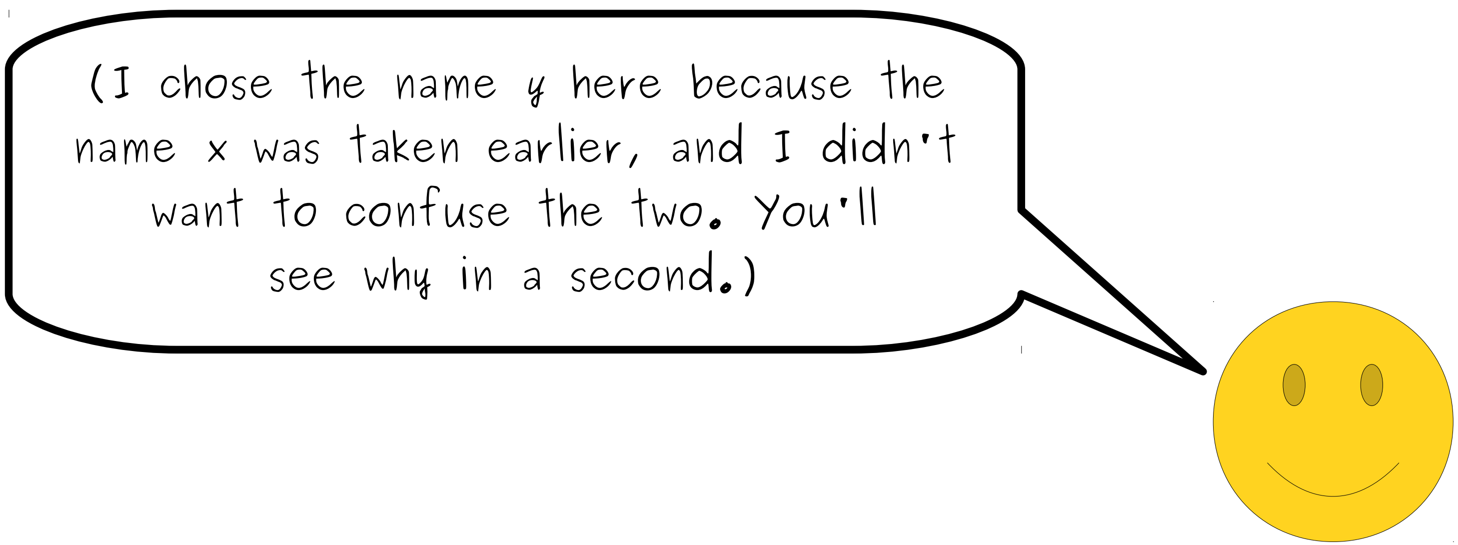

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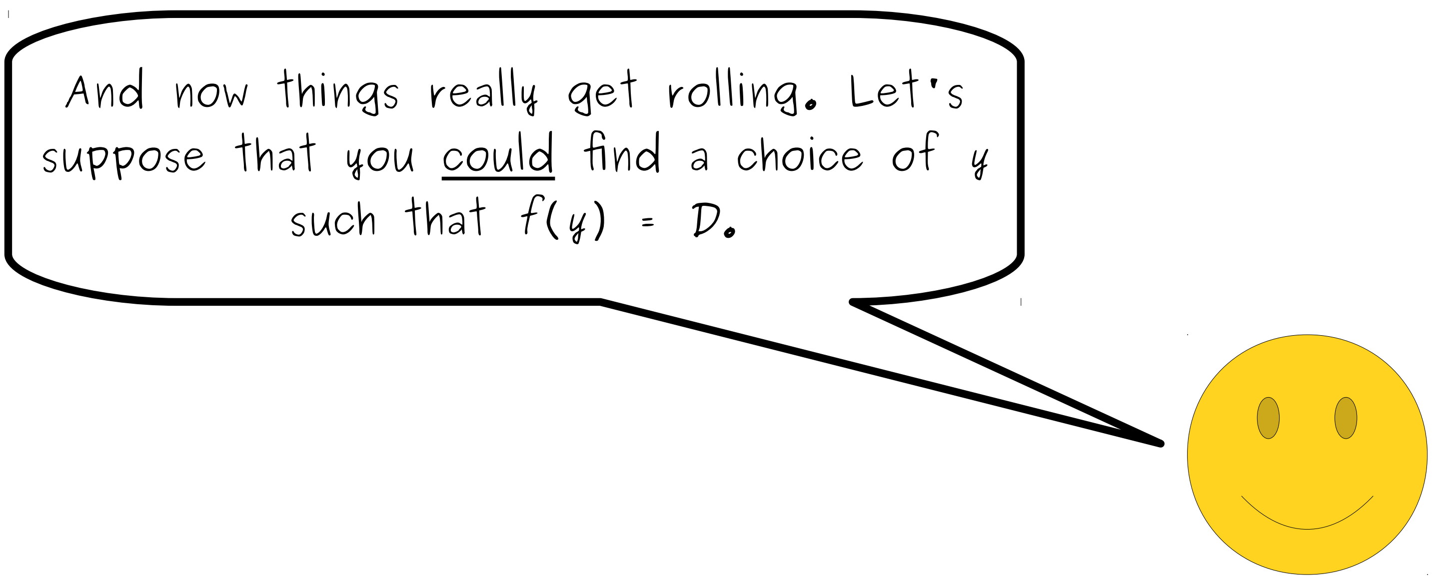

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if $y \notin f(y)$, then $y \in D$, and
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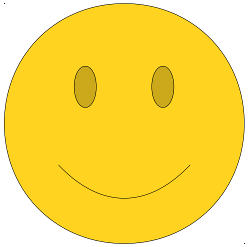
This is just the mathematical way of talking about what happens on the diagonal.



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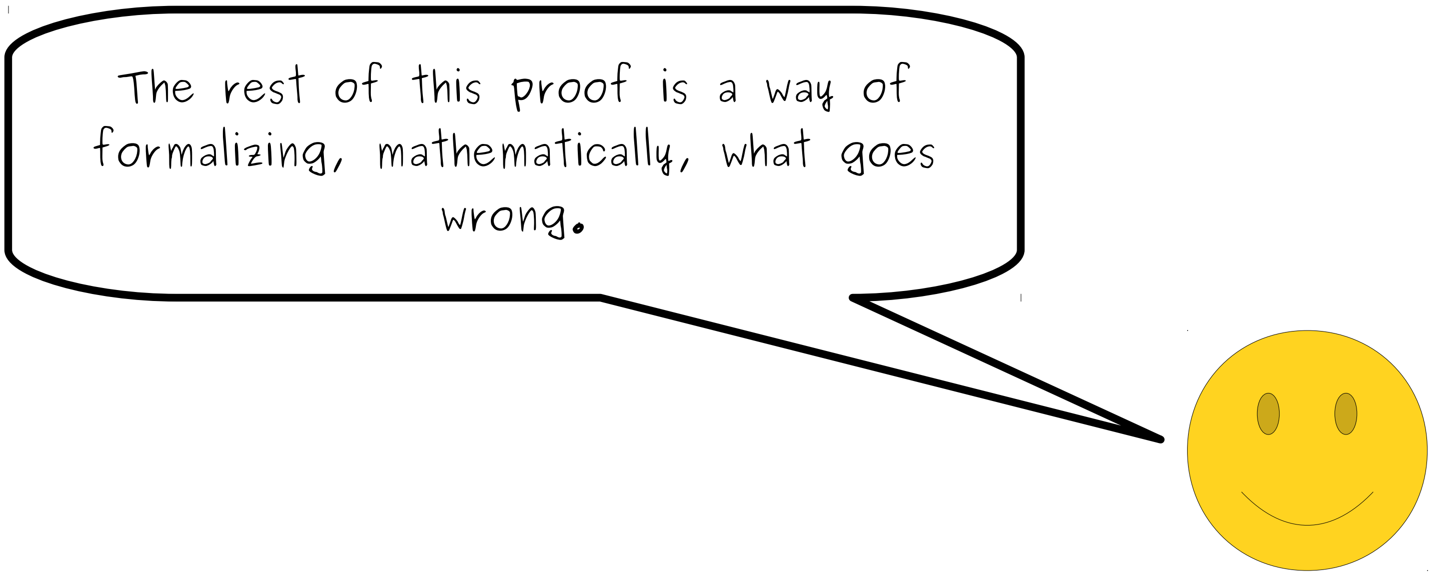

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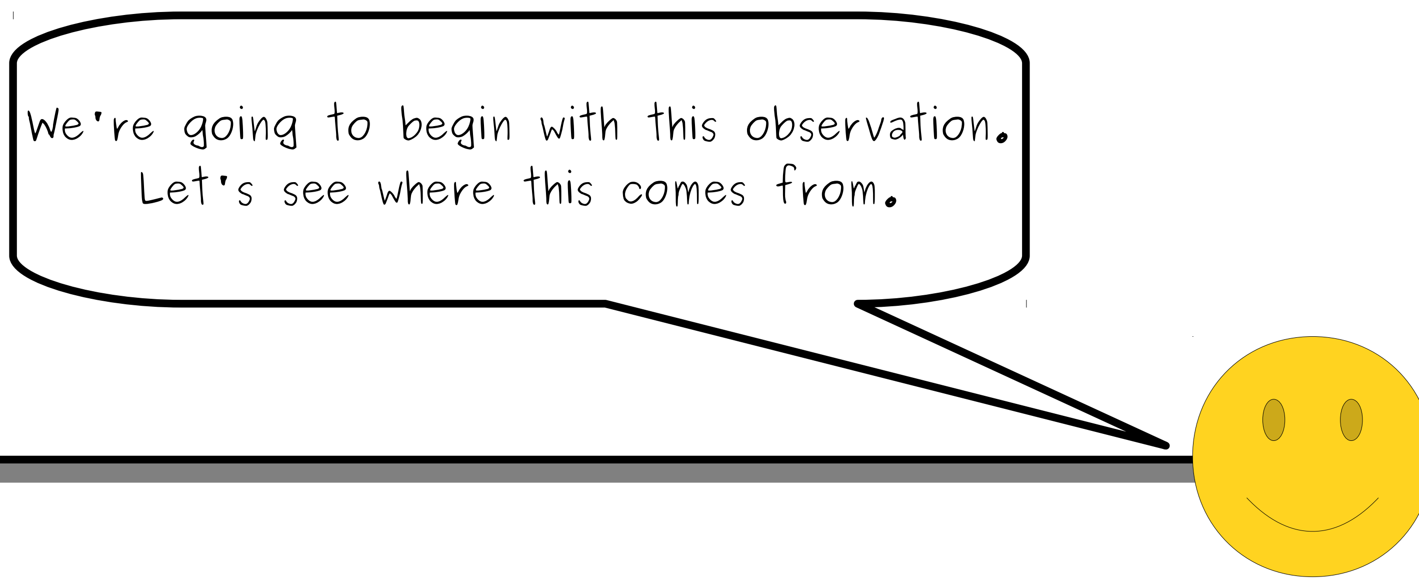

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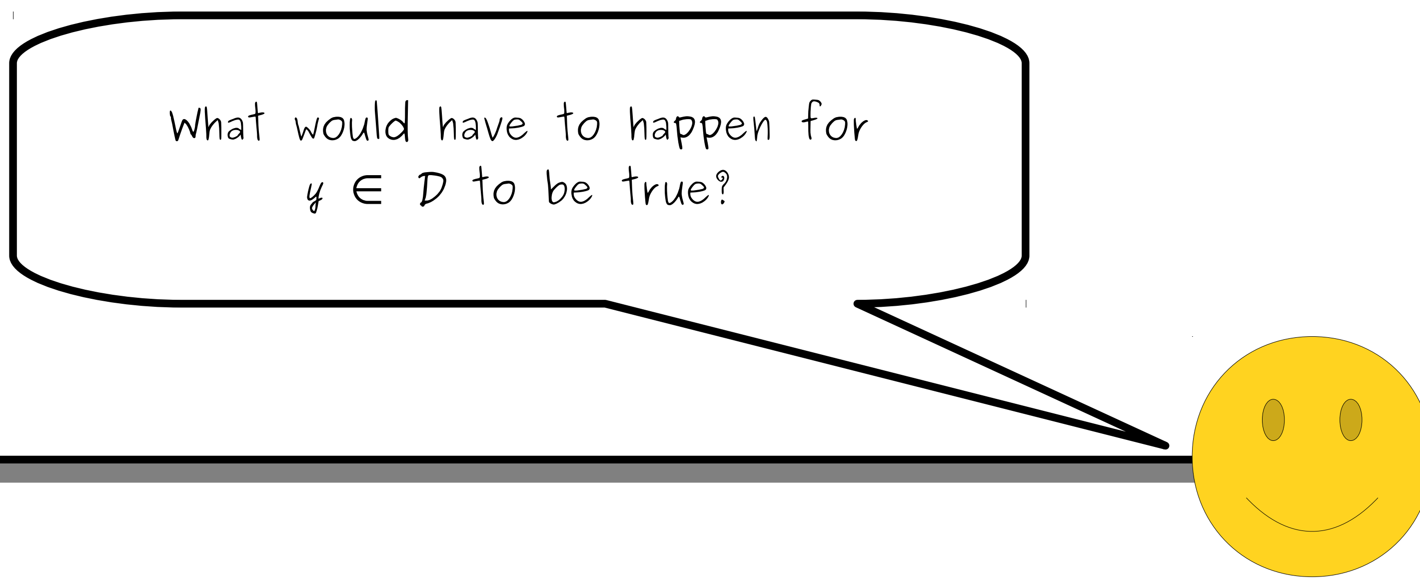

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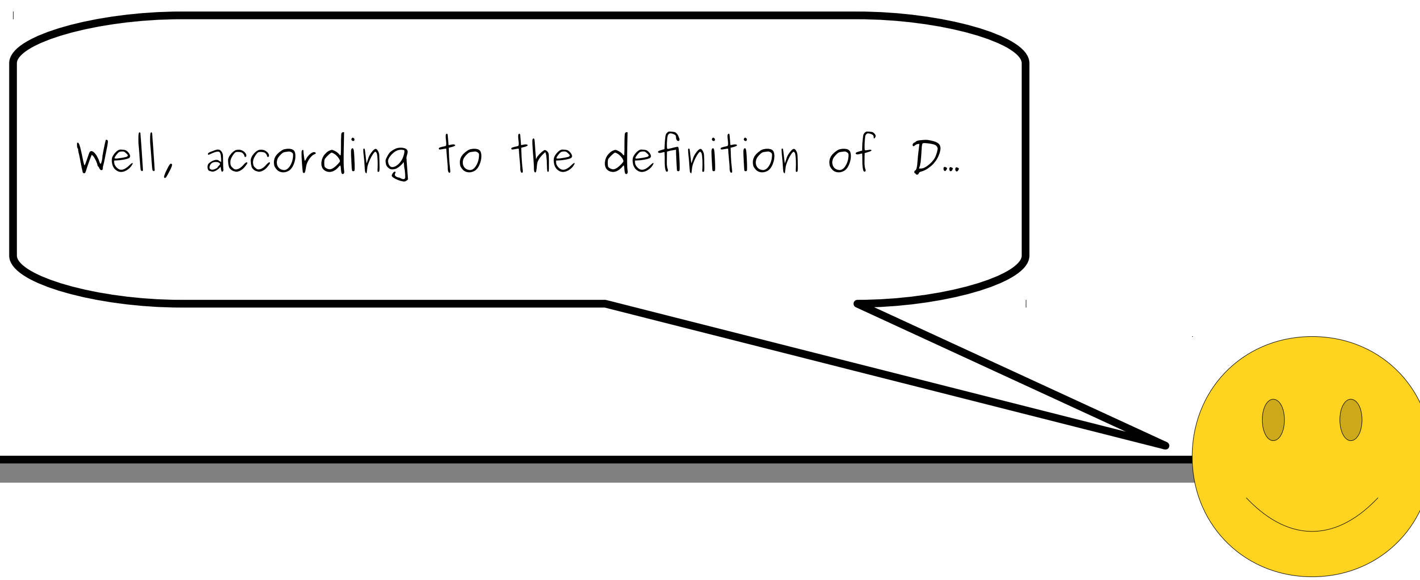

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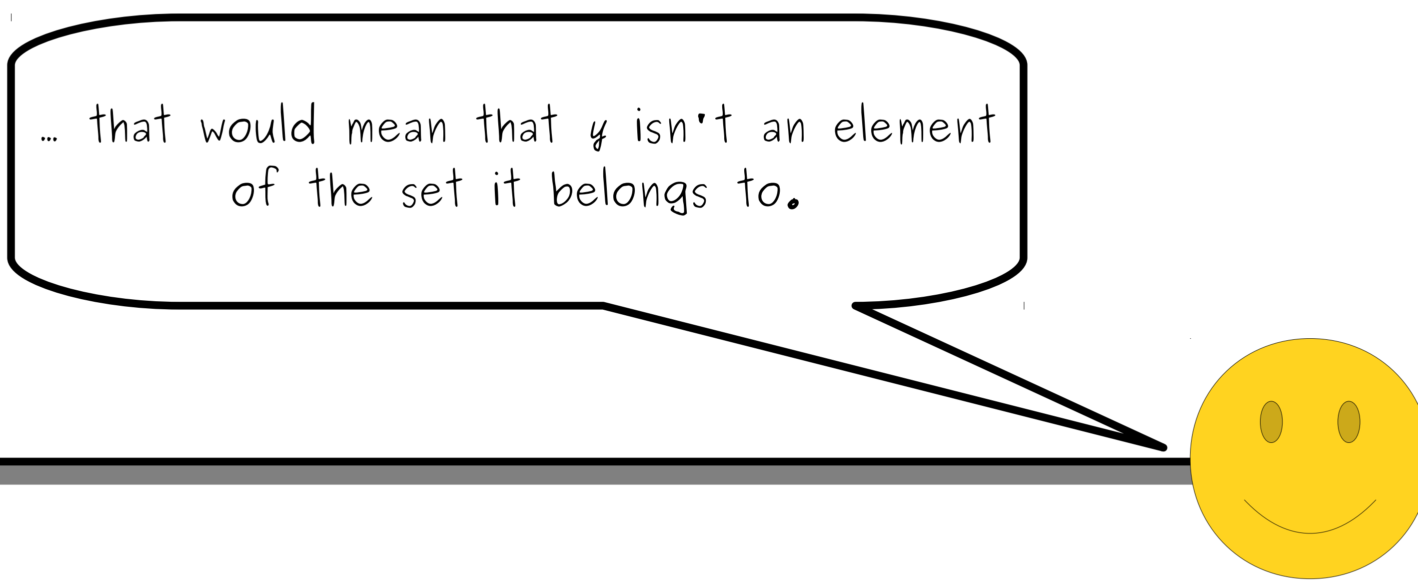

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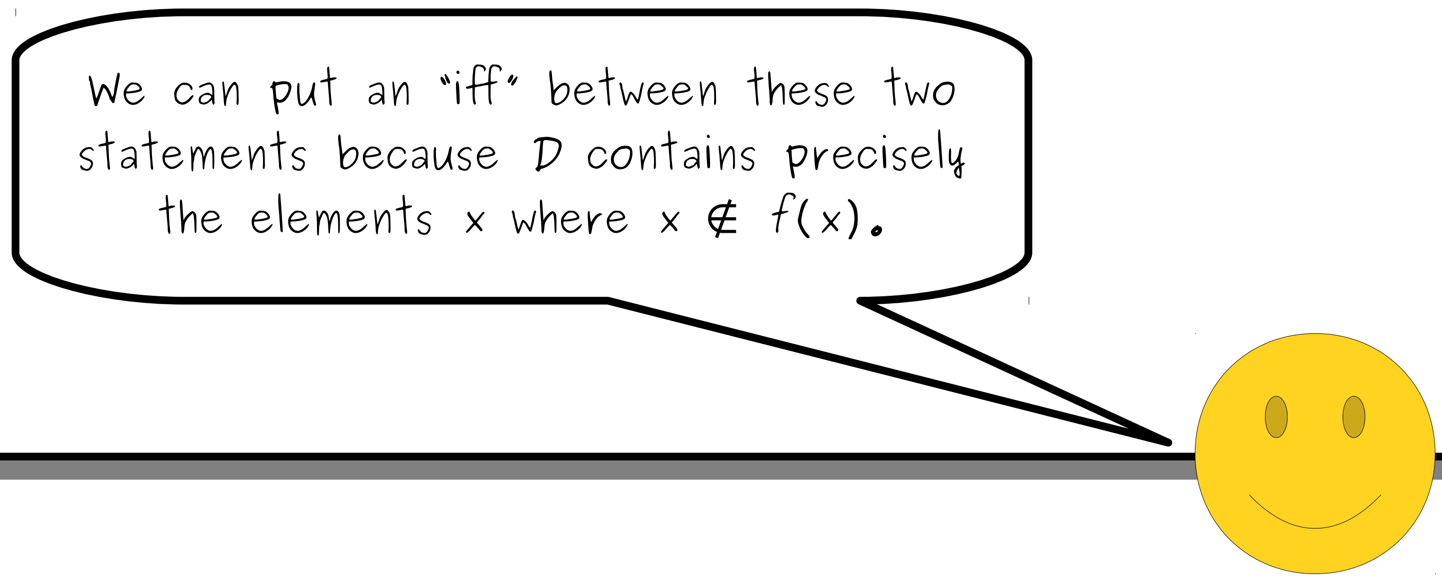

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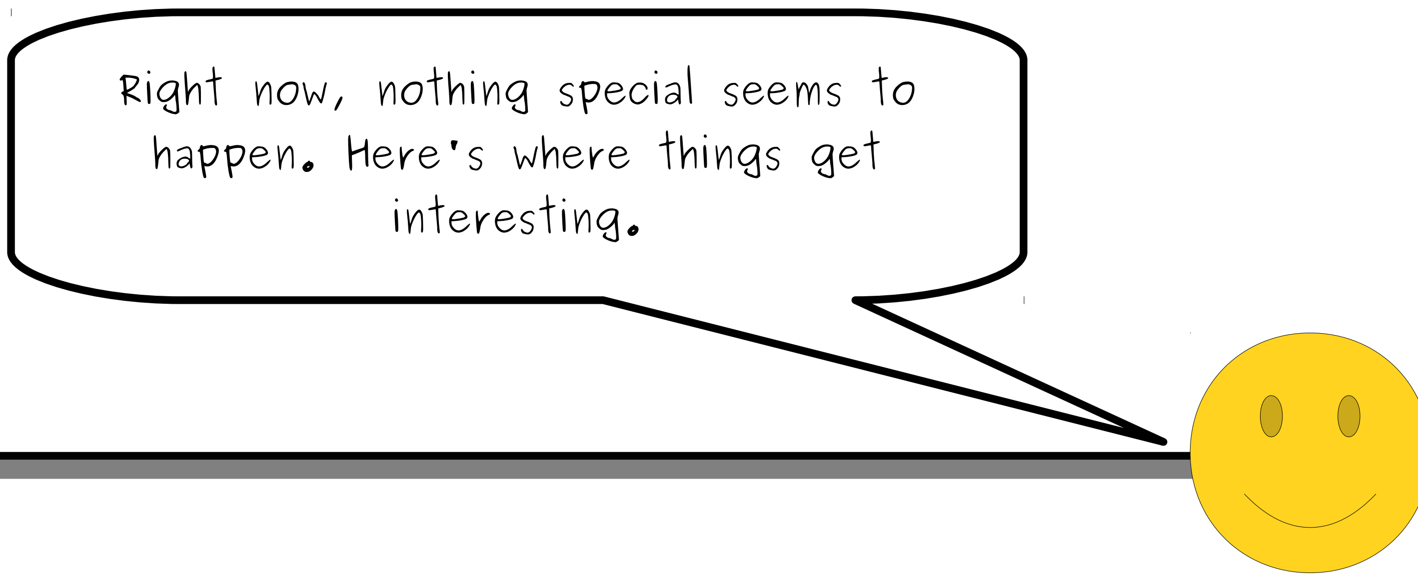

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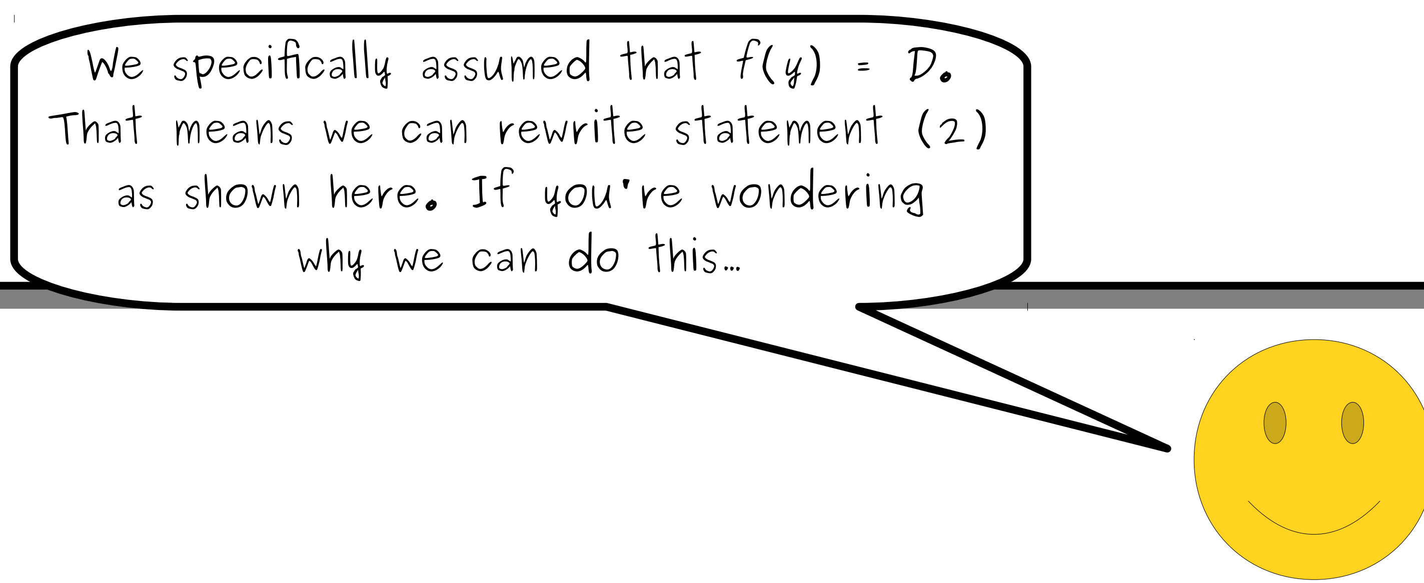

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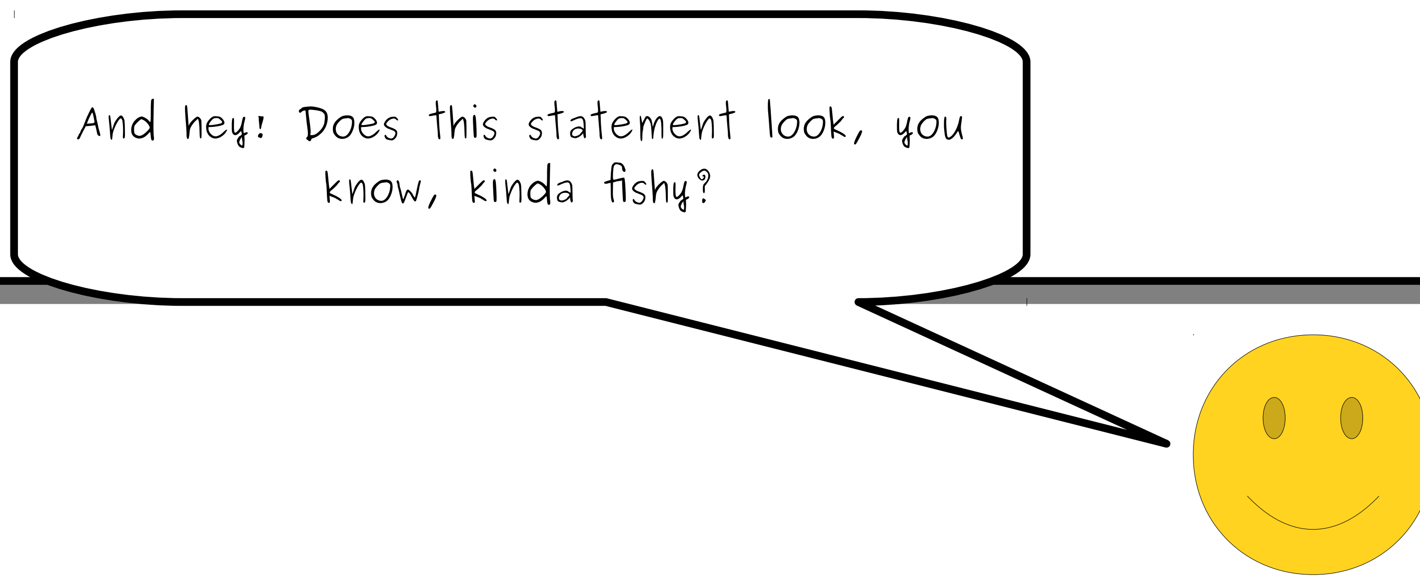

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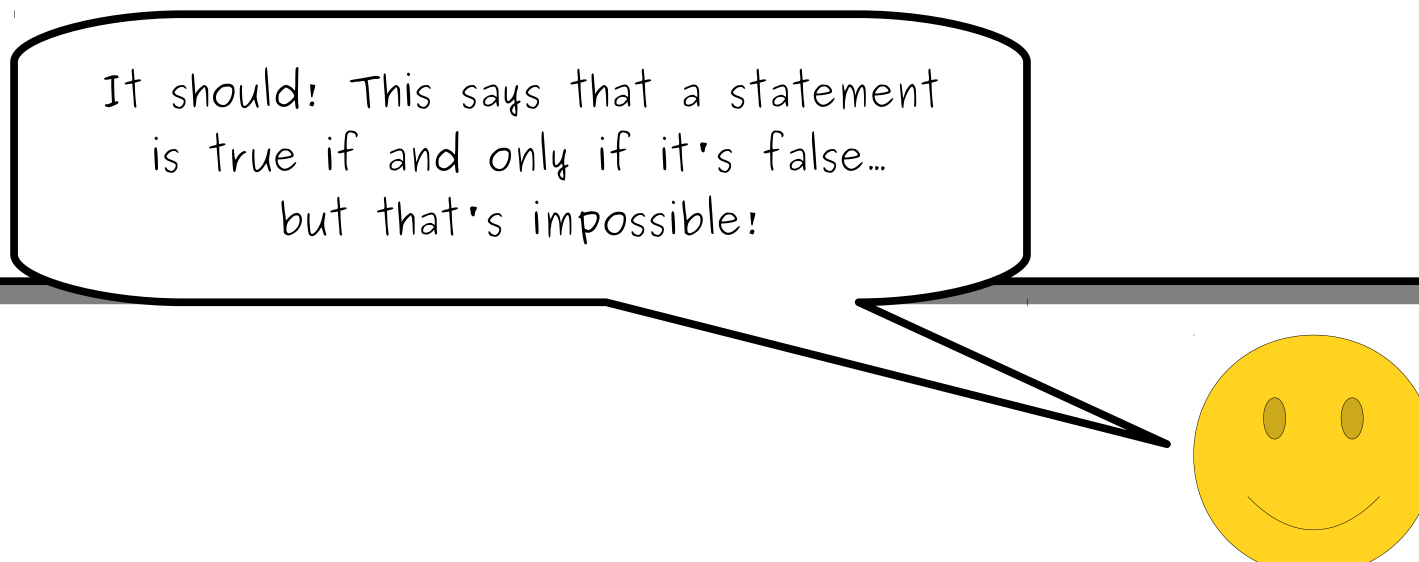

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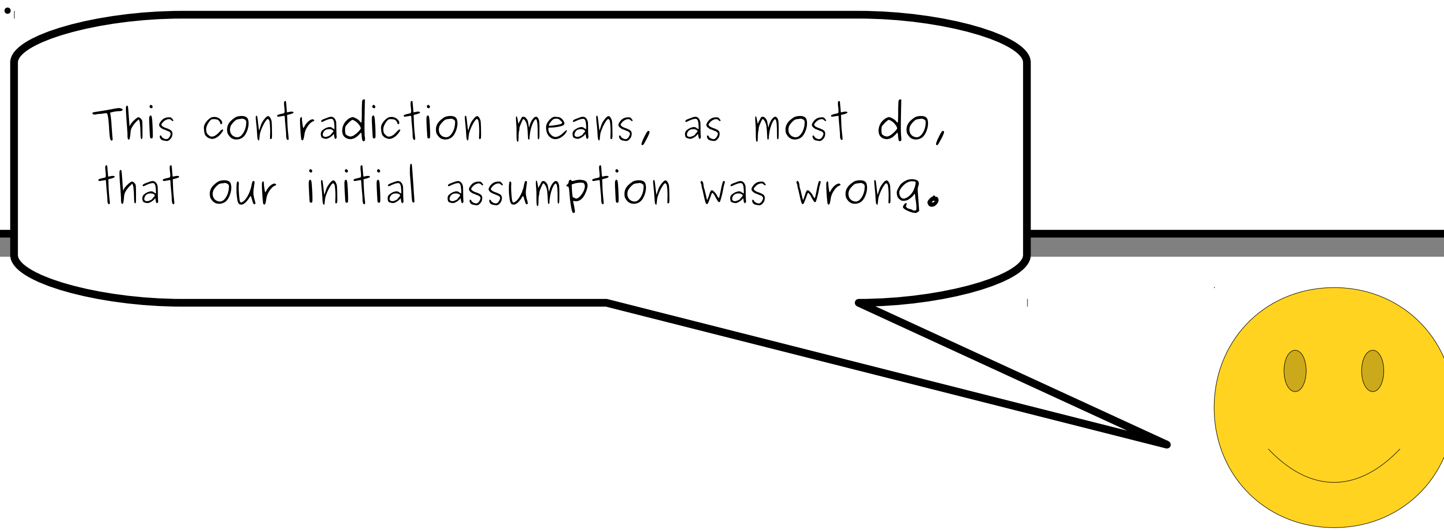

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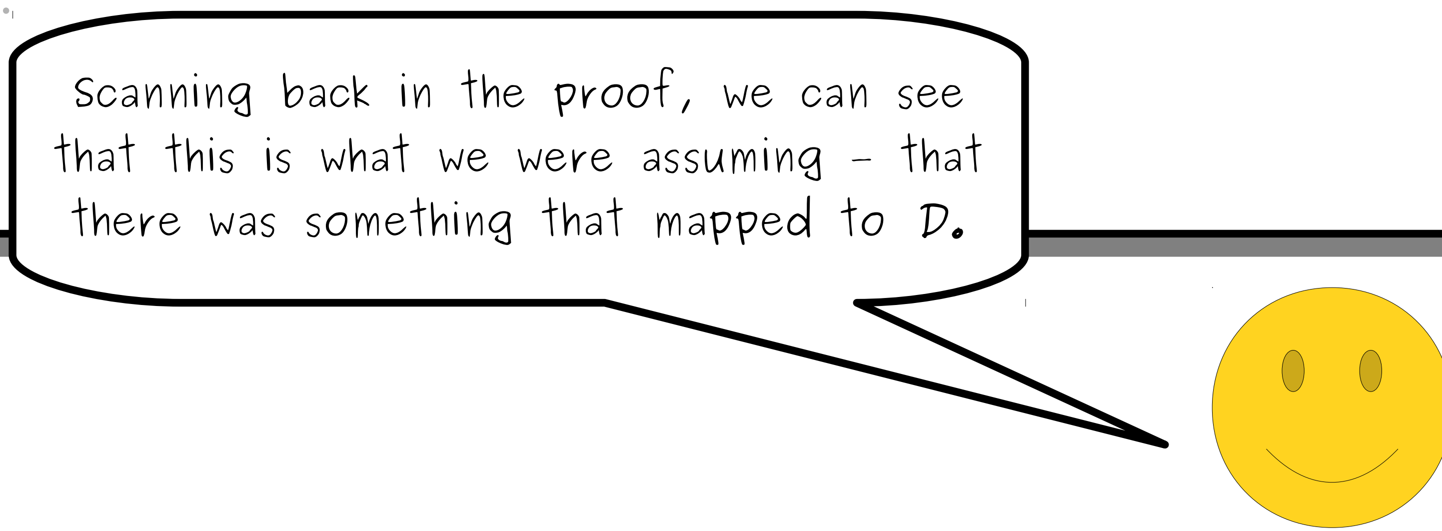

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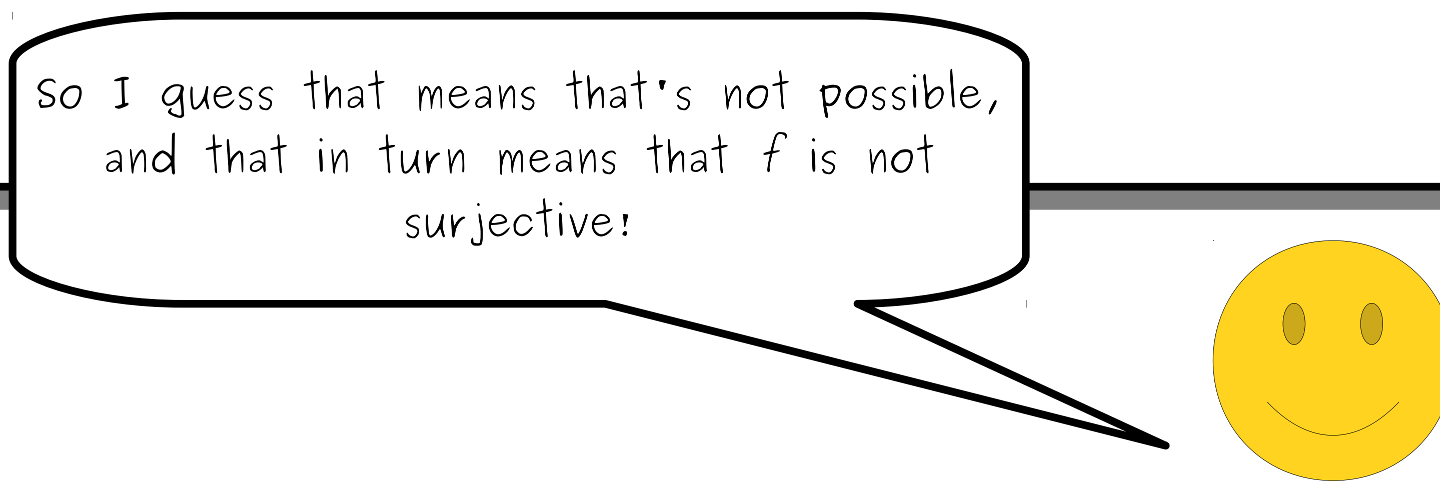

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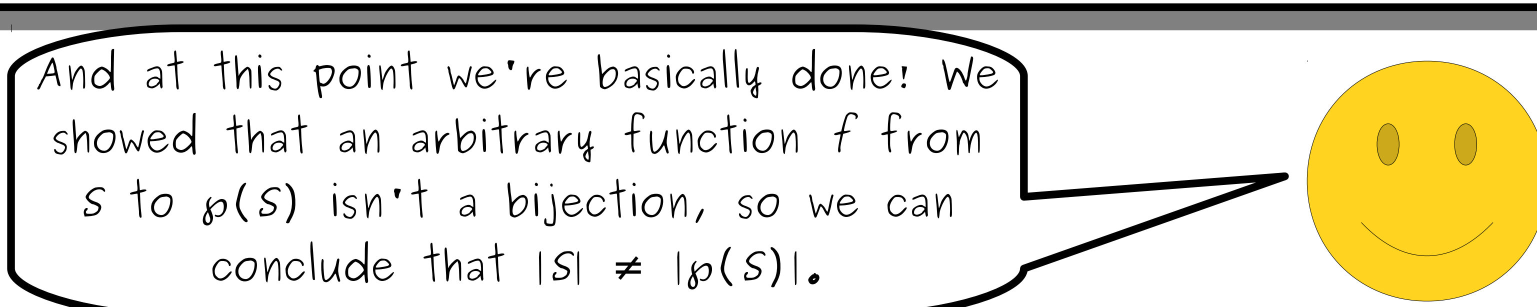

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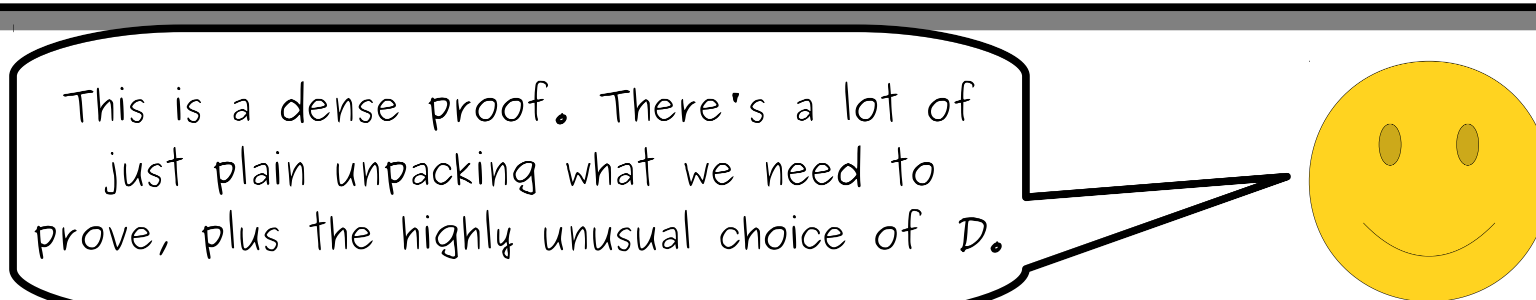

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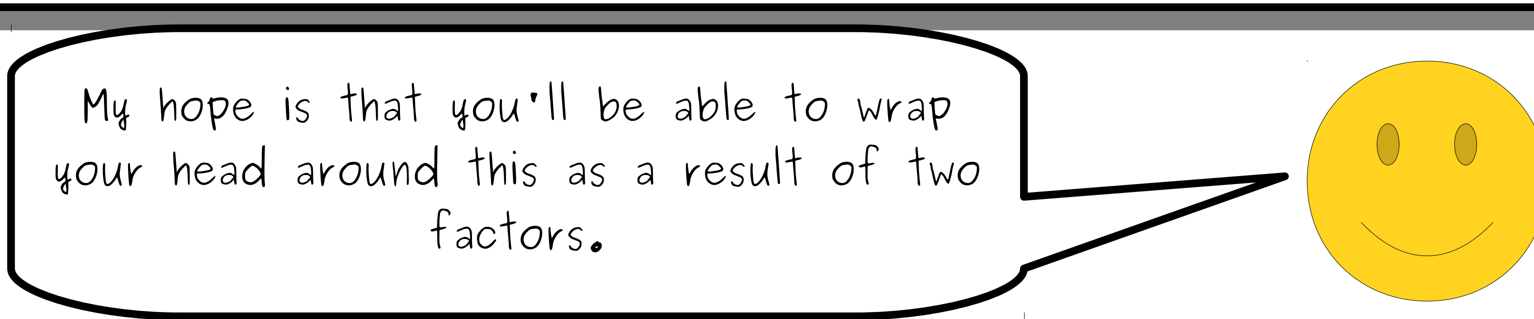

Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. We will prove that f is not surjective.

Starting with f, we define the set

$$D = \{\ x \in S \mid x \notin f(x)\ \}. \tag{1}$$

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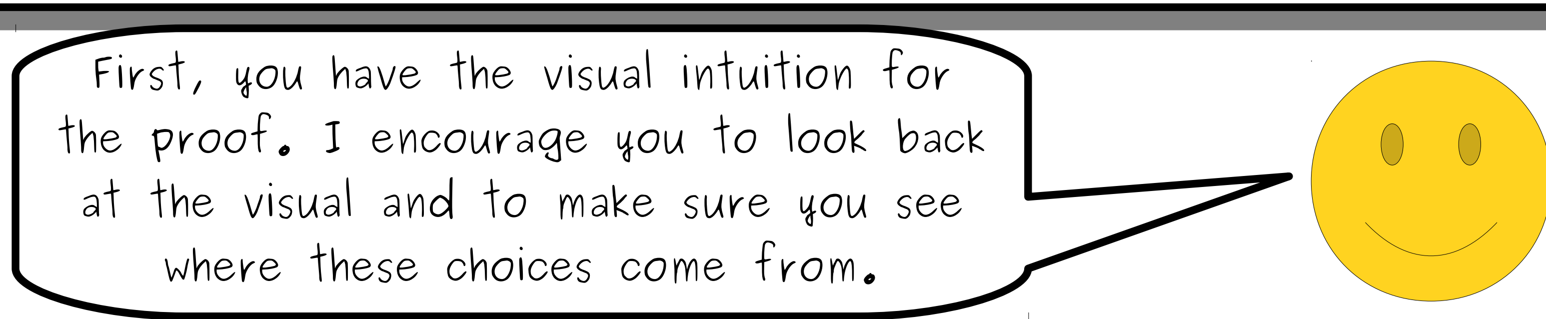

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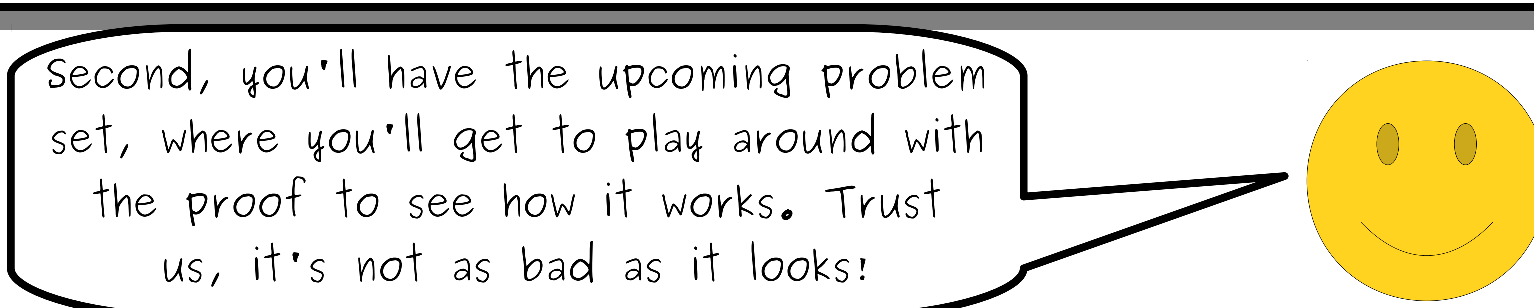



Phew! We've covered a lot of ground here. Let's take stock of what we covered here.

$$\begin{array}{cccccccccc} \mathbb{N} & & & & & 0 & 1 & 2 & 3 & \dots \\ & & & & f : \mathbb{N} \to \mathbb{Z} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{Z} & \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \dots \end{array}$$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

First, we came up with a rigorous definition of what $|S| \leq |T|$ means. This allows us to prove results about how set cardinalities rank against one another… something you (hypothetically speaking) might need for the upcoming problem set.

$$|S| < |T| \quad \textbf{if} \quad |S| \leq |T| \textbf{ and } |S| \neq |T|$$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

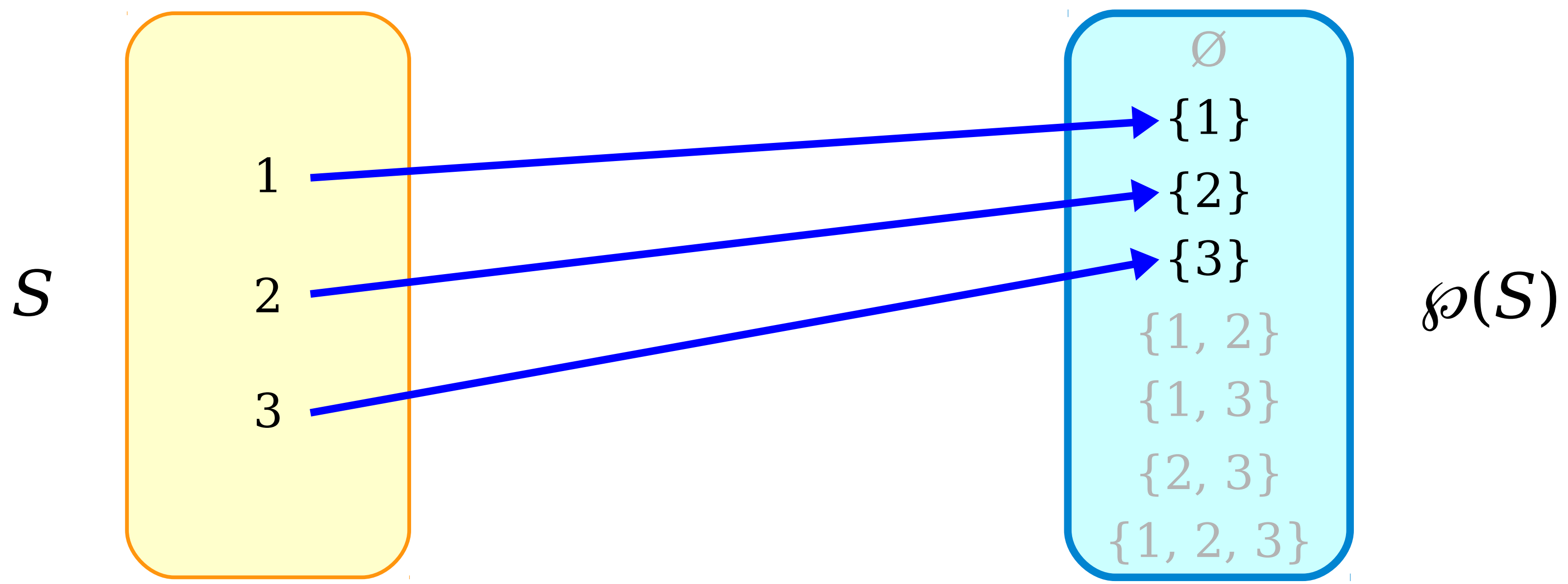
Second, we used that definition to come up with a definition for $|S| < |T|$, which lets us say that one set is strictly smaller than another.

There *is* an injection $f : S \to \wp(S)$
There is *no* bijection $f : S \to \wp(S)$

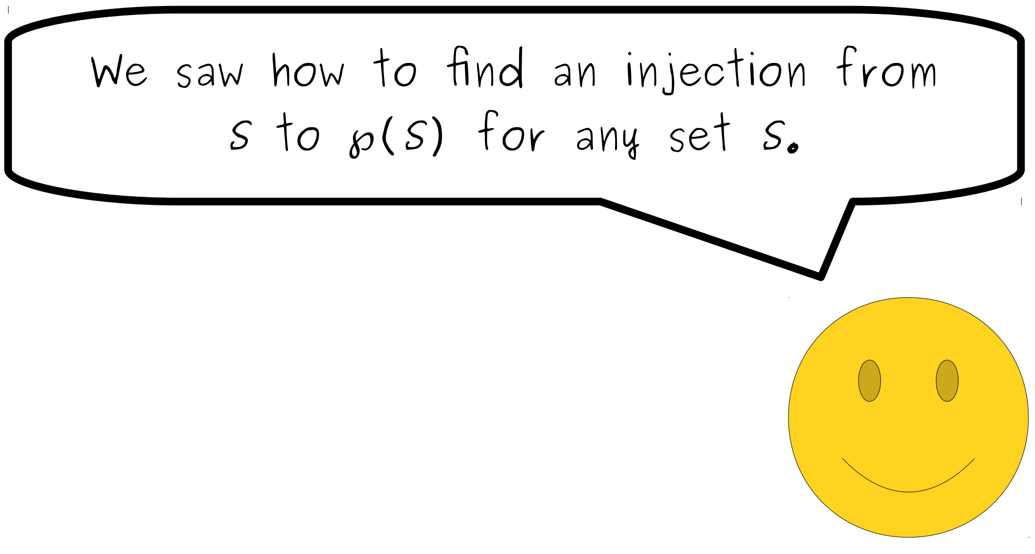
If S is a set, then $|S| < |\wp(S)|$

This in turn allowed us to determine what, specifically, we need to prove to prove Cantor's theorem.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \dots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\ x_4,\quad \dots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \dots\ \}$

$x_3 \longrightarrow \{\quad x_1,\quad x_3,\ x_4,$

$x_4 \longrightarrow \{\ x_0,\quad x_5$

$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5$

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$D = \{\ x \in S \mid x \notin f(x)\ \}$

…

$\{\quad x_1,\ x_2,\quad x_4,\quad \dots\ \}$

Set D

We used the visual intuition of the diagonal argument to precisely define the diagonal set.

Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. We will prove that f is not surjective.

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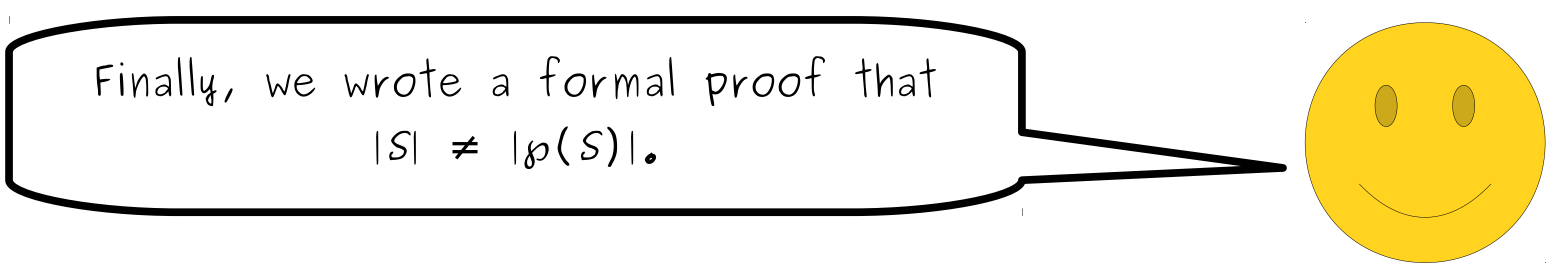
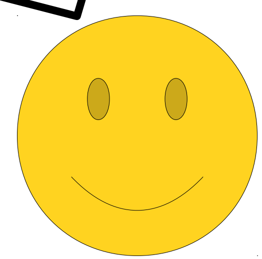
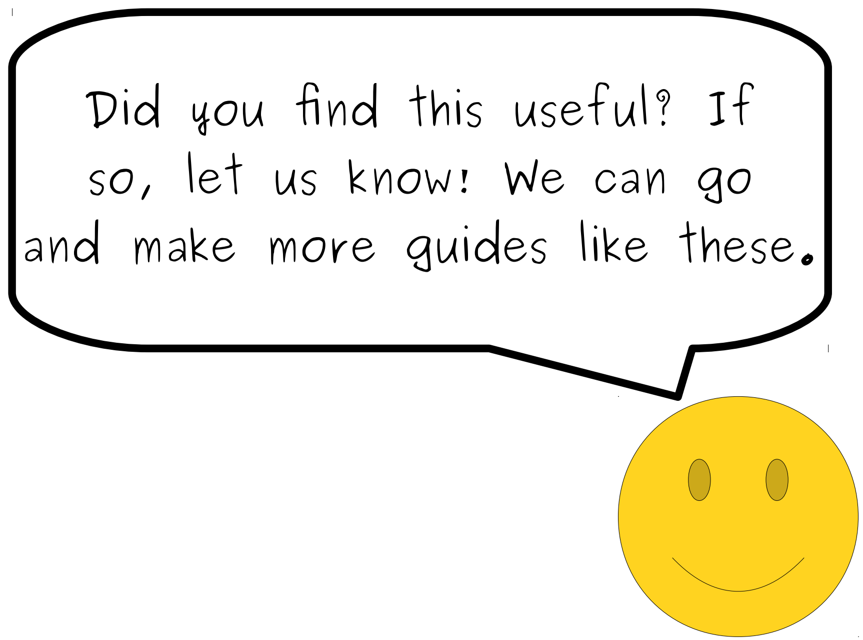


Hope this helps!

Please feel free to ask questions if you have them.

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